# Real Analysis
# Lecture Notes

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# Chapter 1

# $\mathbb{R}$eal number

## 1.1 Field Properties

1. $+$
    - Communitive: $\forall x, y \in \mathbb{R}.\ x + y = y + x$
    - Associative: $\forall x, y, z \in \mathbb{R}.\ (x + y) + z = x + (y + z)$
    - Identity: $\exists 0 \in \mathbb{R}.\ \forall x \in \mathbb{R}.\ x + 0 = x$
    - Additive inverse: $\forall x \in \mathbb{R}.\ \exists - x \in \mathbb{R}.\ x + (-x) = 0$
2. $\cdot$
    - Communitive
    - Associative
    - Identity: $1 \neq 0$
    - Multiplicative inverse: $\exists x^{-1} = \frac{1}{x}.\ x \cdot \frac{1}{x} = 1.$
    - Distributive law: $\forall a, b, c \in \mathbb{R}.\ a \cdot (b + c) = ab + ac$

**Theorem 1.1.1.** *If* $a + x = a$ *then* $x = 0$

*Proof.* Add $-a$ to both side:

$$\begin{aligned} -a + a + x &= -a + a \\ (-a + a) + x &= (-a + a) \\ 0 + x &= 0 \\ x &= 0 \end{aligned}$$

□

**Theorem 1.1.2.** *If* $a + x = 0$ *then* $x = -a$

*Proof.*

$$\begin{aligned} \exists - a \in \mathbb{R}.\ (-a + a) + x &= -a + 0 \\ 0 + x &= -a \\ x &= -a \end{aligned}$$

□

**Theorem 1.1.3.** $\forall a \in \mathbb{R}.\ a \cdot 0 = 0$

*Proof.* Consider

$$\begin{aligned} a + a \cdot 0 &= a \cdot 1 + a \cdot 0 \\ &= a \cdot (1 + 0) \\ &= a \cdot 1 \\ &= a \qquad\qquad \Rightarrow a \cdot 0 = 0 \end{aligned}$$

□

**Theorem 1.1.4.** $\forall a \in \mathbb{R}.\ (-1)a = -a$

*Proof.* Consider

$$\begin{aligned} a + (-1)a &= a \cdot 1 + a(-1) \\ &= a(1 + (-1)) \\ &= a \cdot 0 \\ &= 0 \end{aligned}$$

By theorem 1.1.2

$$(-1)a = -1$$

□

## 1.2 Order

A relation $<$ on $\mathbb{R} \times \mathbb{R}$ satisfying

1. Trichotomy: $\forall a.b \in \mathbb{R}.$ one and only one is true:

$$a = b,\ a < b,\ b < a$$

2. Transitivity: $\forall a, b, c \in \mathbb{R}.$ , if $a < b$ and $b < c$ then $a < c$.

3. Addictive property: $\forall a, b, c \in \mathbb{R}.$ , if $a < b$ then $a + c < b + c$.

4. Multiplicative property: $\forall a, b, c \in \mathbb{R}.$

   (i) if $a < b$ and $c > 0$ then $ac < bc$

   (ii) if $a < b$ and $c < 0$ then $ac > bc$

**Theorem 1.2.1.** $\forall a \in \mathbb{R} \setminus \{0\}\ .\ a^2 > 0$

*Proof.* Since $a \neq 0$, by Trichotomy, $a > 0$ or $a < 0$.

- If $a > 0$, then by $Mp_{(i)}$, $a^2 > a \cdot 0 = 0$, $a^2 > 0$
- If $a < 0$, then by $Mp_{(ii)}$, $a^2 > a \cdot 0 = 0$, $a^2 > 0$

□

**Theorem 1.2.2.** *If $a > 0$ then $a^{-1} = \frac{1}{a} > 0$*

*Proof.* $a^{-1} \neq 0$.

(i) If $a^{-1} = 0$ then $a \cdot a^{-1} = 0$, contradiction.

(ii) If $a^{-1} < 0$, by $Mp_{(ii)}$

$$\begin{aligned} a^{-1} \cdot a &< 0 \\ 1 &< 0 \end{aligned}$$

contradiction.

So by Trichotomy, $a^{-1} > 0$. □

**Theorem 1.2.3.** *If $0 < a < 1$, then $0 < a^2 < a < 1$*

*Proof.* By $Mp_{(i)}$

$$\begin{gathered} 0 \cdot a < a \cdot a < a \cdot 1 \\ 0 < a^2 < a \end{gathered}$$

□

**Definition 1.2.1** (Square root). *For $a > 0$ there is $\sqrt{a} > 0$ such that $(\sqrt{a})^2 = a$*

**Theorem 1.2.4.** *If $0 < a < 1$ then $0 < a < \sqrt{a} < 1$*

*Proof.* First prove $\sqrt{a} < 1$

(i) If $\sqrt{a} > 1$ then $\sqrt{a} > 0$. By $Mp_{(i)}$, $(\sqrt{a})^2 > \sqrt{a} > 1$, $a > 1$, contradiction.

(ii) If $\sqrt{a} = 1$ then $a = (\sqrt{a})^2 = 1$, contradiction.

By Trichotomy, $0 < \sqrt{a} < 1$. By $Mp_{(i)}$

$$\begin{gathered} 0 \cdot \sqrt{a} < \sqrt{a} \cdot \sqrt{a} < \sqrt{a} \cdot 1 \\ 0 < a < \sqrt{a} < 1 \end{gathered}$$

□

**Theorem 1.2.5.** *If $0 \leq a < b$ and $0 \leq c < d$ then $ac < bd$*

*Proof.*

- When $a = 0$ or $c = 0$, $ac = 0$. And $0 < b$ and $0 < d$ so by $Mp_{(i)}$, $0 < bd$, so $ac < bd$.
- Now consider $0 < a < b$ and $0 < c < d$. By $Mp_{(i)}$, $0 < ac < bc$ and $0 < bc < bd$. By transitivity, $ac < bd$.

□

**Theorem 1.2.6.** *If $a > 1$ then $a > \sqrt{a} > 1$*

*Proof.*

$$\begin{gathered} 0 < \frac{1}{a} < 1 \\ 0 < \frac{1}{a} < \sqrt{\frac{1}{a}} < 1 \\ a > \sqrt{a} > 1 \end{gathered}$$

□

**Theorem 1.2.7.** *$a, b \geq 0$ then $\sqrt{a}, \sqrt{b} \geq 0$, then $(\sqrt{a} - \sqrt{b})^2 \geq 0$*

## 1.3 Absolute Value Function

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

**Note.** if $x < 0$, by $Mp_{(ii)}$, $(-1)x > 0$, $-x > 0$.

**Theorem 1.3.1.** $|a| = |-a|$

*Proof.* By Trichotomy

(i) $a > 0$: $|a| = a$, and $-a < 0$, $|-a| = a$

□

**Theorem 1.3.2.** $|ab| = |a|\,|b|$

**Theorem 1.3.3** (Fundamental Theorem Of Absolute Values). $|a| < M \iff -M < a < M$

*Proof.*

1. Assume $|a| < M$, to prove that $-M < a < M$.
   - if $a \geq 0$, then $|a| = a$, $0 \leq a < M$. Since $0 < M$, $-M < 0$, $-M < a < M$.
   - if $a < 0$, then $|a| = -a$, $0 < -a < M$, $M > 0 > a > -M$, so $-M < a < M$.
2. Assume $-M < a < M$, to prove that $|a| < M$.
   - if $a \geq 0$ then $|a| = a < M$.
   - if $a < 0$ then $|a| = -a$. Since $M > -a > -M$, $M > |a|$.

□

**Theorem 1.3.4** (1st Triangle Inequality). $|a + b| \leq |a| + |b|$

*Proof.*

$$\begin{gathered} |a| \leq |a| \\ -|a| \leq a \leq |a| \\ |b| \leq |b| \\ -|b| \leq b \leq |b| \\ -|a| - |b| \leq a + b \leq |a| + |b| \\ |a + b| \leq |a| + |b| \end{gathered}$$

□

**Theorem 1.3.5** (2nd Triangle Inequality). $||a| - |b|| \leq |a - b|$

*Proof.*

$$\begin{gathered} |a| = |a - b + b| \leq |a - b| + |b| \\ |a| - |b| \leq |a - b| \\ |b| = |b - a + a| \leq |b - a| + |a| \\ -|b - a| \leq |a| - |b| \\ ||a| - |b|| \leq |a - b| \end{gathered}$$

□

**Theorem 1.3.6.** *If* $|a| < \epsilon$ *for* $\epsilon > 0$ *then* $a = 0$

*Proof.* Suppose $|a| > 0$. set $\epsilon = \frac{|a|}{2} > 0$, then $|a| > \epsilon$. Contradiction. So $|a| = 0$, $a = 0$. □

## 1.4 Supremum and Infimum

Let $E \subseteq \mathbb{R}, E \neq \varnothing$.

**Definition 1.4.1** (Bounded above). *$\exists M \in \mathbb{R}. \forall x \in E. x < M$. $M$ is the upper bound.*

**Definition 1.4.2** (Supremum).

*(i) $\forall x \in E. x \leq \sup E$*

*(ii) If $M$ is a upper bound of $E$, then $\sup E \leq M$ (Or, no $M \leq \sup E$ is an upperbound)*

**Definition 1.4.3** (Bounded below). *$\exists M \in \mathbb{R}. \forall x \in E. x > M$. $M$ is the lower bound.*

**Definition 1.4.4** (Infimum).

*(i) $\forall x \in E. x \geq \inf E$*

*(ii) If $M$ is a lower bound of $E$, then $\inf E \geq M$*

**Ex 1.4.1.** $E = [0, 1] = \{x \mid 0 \leq x \leq 1\}, \sup E = 1, \inf E = 0$

**Ex 1.4.2.** $E = (0, 1) = \{x \mid 0 < x < 1\}, \sup E = 1, \inf E = 0$

*Proof.* Show if $M < 1$ then it's not an upper bound, therefore all upperbound of $E$ greater or equal to 1. If $\frac{1}{2} < M < 1$ then

$$M = \frac{M + M}{2} < \frac{M + 1}{2} < \frac{1 + 1}{2} = 1$$

We have $\frac{M+1}{2} \in E$, $\frac{M+1}{2} > M$, $M$ is not an upper bound. Therefore all upper bounds $M$ must be $\geq 1$. So, by def, $\sup E = 1$. $\square$

**Theorem 1.4.1.** *If $s = \sup E$ and $r = \sup E$ then $s = r$*

*Proof.* $s \leq$ all upper bounds, $r$ is an upper bound, $s \leq r$. $r \leq$ all upper bounds, $s$ is an upper bound, $r \leq s$. Therefore, by Trichotomy, $s = r$. $\square$

**Theorem 1.4.2.** *If $a \in E$, and $a$ is an upperbound for $E$ then $\sup E = a$.*

*Proof.* $a$ satisfies (i) for being a sup. Since $a \in E$. If $M$ is a upperbound of $E$, $a < M$. $a$ satisfies (ii) for being a sup. So $a = \sup E$. $\square$

**Definition 1.4.5.** *$M$ is **not** an upperbound for $E$ means $\exists x \in E. x > M$.*

**Theorem 1.4.3.** *For $E$, $\sup E$ exists, $\epsilon > 0$.*

$$\sup E - \epsilon < \sup E$$

*So $\sup E - \epsilon$ is not an upperbound, meaning $\exists x \in E.\ \sup E - \epsilon < x \leq \sup E$.*

**Ex 1.4.3.** *Let $A$ be an nonempty, bounded set, $c > 0$. $B = \{x = ca, a \in A\}$. Prove that $\sup B = c \cdot \sup A$*

*Proof.* By compeleteness, $\sup A$ exists. $\forall x \in B.\ , \exists a \in A.\ x = c{\cdot}a$. Since $a \le \sup A$ we have $x = ca \le c{\cdot}\sup A$. So $c \cdot \sup A$ is an upperbound of $B$. By compeleteness, $\sup B$ exists. Follows that $\sup B \le c \cdot \sup A$. Now, since $\sup B$ is an upperbound of $B$,

$$\begin{aligned} \forall x \in B.\ x &\le \sup B \\ \forall a \in A.\ ca &\le \sup B \\ a &\le \frac{\sup B}{c} \end{aligned}$$

So $\frac{\sup B}{c}$ is an upperbound for $A$, entails $\frac{\sup B}{c} \ge \sup A$, namely $\sup B \ge c \cdot \sup A$. So $\sup B = c \cdot \sup A$. $\square$

**Ex 1.4.4.** *Let $A, B$ be nonempty, bounded sets. What is* $\sup(A - B)$

$$\begin{aligned} \sup(A - B) &= \sup(A + (-B)) \\ &= \sup(A) + \sup(-B) \\ &= \sup(A) - \inf(B) \end{aligned}$$

## 1.5 Completeness

**Definition 1.5.1.** *If $E \subseteq \mathbb{R}$, $E \neq \varnothing$ and $E$ is bounded above then $\sup E$ exists. (is a real number)*

**Ex 1.5.1.** *For rational number:*

$$E = \left\{ \frac{n}{m} \in \mathbb{Q} \mid \frac{n}{m} < \pi \right\}, \sup E = \pi \notin \mathbb{Q}$$

*So $\mathbb{Q}$ is not complete.*

**Definition 1.5.2** ($\sup \mathbb{Z}$)**.** *if $E \subseteq \mathbb{Z} \subseteq \mathbb{R}$, and $\sup E$ exists, then $\sup E \subseteq E$.*

## 1.6 Archimedean Principle (AP)

**Definition 1.6.1.** *For all $a, b \in \mathbb{R}$, $a > 0$, there is an $N \subseteq \mathbb{N}$ s.t. $Na > b$.*

*Proof.*

1. If $a > b$, then $N = 1$
2. If $a \le b$ then let
$$E = \{k \in \mathbb{N} \mid ka \le b\}$$
Since $a \le b$, $k = 1 \in E$, so $E$ is not empty, $k \in E \Rightarrow k \le \frac{b}{a}$, $\frac{b}{a}$ is an upper bound of $E$. By Completeness, $\sup E$ exists. Call $n = \sup E$. By $\sup \mathbb{Z}$, $n \in E$. Now $n + 1$ is not in $E$, therefore $(n + 1)a > b$. Set $N = n + 1$.

$\square$

## 1.7 Density of $\mathbb{Q}$ in $\mathbb{R}$

**Definition 1.7.1** (Density). $\forall a, b \in \mathbb{R}.\ a < b, \exists r \in \mathbb{Q}.\ a < r < b.$

*Proof.* By A.P. then there is an $N \subseteq \mathbb{N}$ s.t. $\frac{1}{N} < b - a$. Let

$$E = \left\{ k \in \mathbb{Z} \mid \frac{k}{n} \leq a \right\}$$

$E$ is nonempty, bounded above by $Na$. By Completeness, $\sup E$ exists. By $\sup \mathbb{Z}$, $\sup E \in E$. Set $n = \sup E$, then

$$\begin{aligned}
n + 1 &\notin E \\
\frac{n+1}{n} &> a \\
\frac{n}{n} \leq a &< \frac{n+1}{n} \\
&= \frac{n}{n} + \frac{1}{n} < a + b - a = b \\
a < \frac{n+1}{n} &< b
\end{aligned}$$

let $r = \frac{n+1}{n} \in \mathbb{Q}$. ◻

## 1.8 Reflection

**Definition 1.8.1** ($-E$). *$E \subseteq R$. Let $-E = \{a \mid a = -x, x \in E\}$.*

**Theorem 1.8.1.** *If $\sup E$ exists, then $\inf(-E)$ exists, and equals to $-\sup E$.*

*Proof.* Since $\sup E$ exists,

$$\begin{aligned}
\forall x \in E.\ x &\leq \sup E \\
\forall x \in E.\ -x &> -\sup E \\
\forall a \in -E.\ a &\geq -\sup E
\end{aligned}$$

So $-\sup E$ is a lower bound for $-E$.

$$\begin{aligned}
\forall a \in -E.\ a &\geq M \\
\forall a \in -E.\ -a &\leq -M \\
\forall x \in E.\ x &\leq -M
\end{aligned}$$

Therefore $\sup E \leq -M$, $-\sup E \geq M$. ◻

## 1.9 Monotonicity

**Theorem 1.9.1.** *If $A \subseteq B$, $A \neq \varnothing$, $\sup B$ exists, then $\sup A \leq \sup B$.*

*Proof.* If $a \in A \subseteq B$ then $a \in B$, $a \leq \sup B$. $\sup B$ is an upperbound for $A$. By Completeness $\sup A$ exists and $\sup A \leq \sup B$. □

**Theorem 1.9.2.** *If $A \subseteq B$, $A \neq \varnothing$, and $\inf B$ exists, then $\inf A \geq \inf B$.*

**Definition 1.9.1** (Sup and Inf of any set)**.** *Let $E \subseteq \mathbb{R}$.*

- *For $E \neq \varnothing$, If $E$ is not bounded above by any number, then $\sup E = \infty$. If $E$ is not bounded below by any number, then $\inf E = -\infty$.*
- *For $E = \varnothing$, $\sup E = -\infty$, $\inf E = \infty$.*

# Chapter 2

# Sequences on $\mathbb{R}$

## 2.1 Limits of Sequences

**Definition 2.1.1** (Limit)**.**

$$\lim_{n\to\infty} x_n = L \iff \forall\epsilon > 0.\ \exists N.\ n \geq N \to |x_n - L| < \epsilon$$

**Theorem 2.1.1.** *If $x_n \to L$ as $n \to \infty$ then all subsets also converge to $L$.*

**Theorem 2.1.2.** *If $x_n \to L$ as $n \to \infty$ then $\{x_n\}$ is bounded*

$$\forall n \in \mathbb{N}.\ \exists M.\ |x_n| \leq M$$

**Ex 2.1.1.** *True or False. If $x_n$ converges then $\frac{x_n}{n}$ converges.*

**Answer.** True. $\lim_{n\to\infty} \frac{x_n}{n} = 0$.

*Proof.* Consider, since $\forall n \in \mathbb{N}.\ |x_n| \leq M$, without lost of generality, $M > 0$.

$$\frac{|x_n|}{n} \leq \frac{M}{n}$$

Given $\epsilon > 0$. By A.P, $\exists N \in \mathbb{N}.$ so that $M < N\epsilon$.

$$\forall n \geq M.\ \left|\frac{x_n}{n} - 0\right| \leq \frac{M}{n} \leq \frac{M}{N} < \epsilon$$

□

**Ex 2.1.2.** *True or False. If $x_n$ does not converge, then $\frac{x_n}{n}$ does not converge.*

**Answer.** False. Consider $x_n = (-1)^n$. $x_n$ does not converge but $\frac{x_n}{n} \to 0$ as $n \to \infty$.

*Proof.* Given $\epsilon > 0$. By A.P, $\exists N \in \mathbb{N}.\ 1 < N\epsilon$. For $n \geq N$ we get

$$\left|\frac{(-1)^n}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \epsilon$$

□

**Theorem 2.1.3.** $\frac{\pi}{\sqrt{n}} \to 1$ *as* $n \to \infty$

*Proof.* Given $\epsilon > 0$. By A.P., $\exists N \in \mathbb{N}$. $\pi^2 < N \cdot \epsilon^2$.

$$\frac{\pi^2}{N} < \epsilon^2$$
$$\frac{\pi}{\sqrt{N}} < \epsilon$$

For all $n \geq N$ we get

$$\left|1 + \frac{\pi}{\sqrt{n}} - 1\right| = \frac{\pi}{\sqrt{n}} \leq \frac{\pi}{\sqrt{N}} < \epsilon$$

□

**Theorem 2.1.4.** *Assume that $x_n \to 1$ as $n \to \infty$, then*

$$\lim_{n\to\infty} \frac{\pi x_n - 2}{x_n} = \pi - 2$$

*Proof.* By assumption, take $\epsilon = \frac{1}{2}$, there exists $N$ so that $\forall n \geq N$. $|x_n - 1| \leq \frac{1}{2}$, gives $\frac{1}{2} < x_n < \frac{3}{2}$.

$$\begin{aligned}
\left|\frac{\pi x_n - 2}{x_n} - (\pi - 2)\right| &= \left|\frac{\pi x_n - 2}{x_n} - \frac{(\pi-2)x_n}{x_n}\right| \\
&= \left|\frac{2x_n - 2}{x_n}\right| \\
&= \frac{2}{|x_n|}\,|x_n - 1| \\
&\leq 4\,|x_n - 1|
\end{aligned}$$

By assumption, for any $\epsilon > 0$, $|x_n - 1| < \frac{\epsilon}{4}$

$$\left|\frac{\pi x_n - 2}{x_n} - (\pi - 2)\right| < \epsilon$$

□

**Theorem 2.1.5** (Comparison)**.** *If $x_n \to x$ and $y_n \to y$ as $n \to \infty$ and if $x_n \leq y_n$ for $n \geq N_0$ then $x \leq y$.*

*Proof.* Suppose not, $x > y$. For $\epsilon = \frac{x-y}{2} > 0$, then there is $N_1$ s.t. $n \geq N_1 \to |x_n - x| < \frac{x-y}{2}$ and $N_2$ s.t. $n \geq N_2 \to |y_n - y| < \frac{x-y}{2}$.

$$\begin{aligned}
\frac{x+y}{2} &= x - \frac{x-y}{2} < x_n \\
y_n &< \frac{x-y}{2} + y = \frac{x+y}{2} \\
y_n &< \frac{x+y}{2} < x_n \text{ for } n > \max(N_1, N_2)
\end{aligned}$$

□

**Ex 2.1.3.** *True or False. If $x_n \to \infty$ as $n \to \infty$ then $\frac{1}{x_n} \to \infty$.*

**Answer.** False. $x_n = -\frac{1}{n}$

**Theorem 2.1.6.** *If $x_n \geq 0$ and $x_n \geq 0$ as $n \to \infty$ then $\sqrt{x_n} \to \sqrt{x}$ as $n \to \infty$.*

*Proof.* Since $x_n \geq 0$ by comparison $x \geq 0$.

- If $x = 0$, consider $\left|\sqrt{x_n} - 0\right| = \sqrt{x_n}$ since $x_n \to 0$, given $\epsilon > 0$ there is an $N$ so that $n \geq N \to |x_n - 0| < \epsilon^2$. $x_n < \epsilon^2$, $\sqrt{x_n} < \epsilon$, $\left|\sqrt{x_n} - 0\right| < \epsilon$.

- If $x > 0$, consider

$$\begin{aligned}\left|\sqrt{x_n} - \sqrt{x}\right| &= \left|(\sqrt{x_n} - \sqrt{x})\frac{\sqrt{x_n}+\sqrt{x}}{\sqrt{x_n}+\sqrt{x}}\right| \\ &= \frac{|x_n - x|}{\sqrt{x_n}+\sqrt{x}} \\ &\leq \frac{|x_n - x|}{\sqrt{x}}\end{aligned}$$

Given $\epsilon > 0$ then there is an $N$ s.t.

$$n \geq N \to |x_n - x| < \sqrt{x}\epsilon$$

So

$$\left|\sqrt{x_n} - \sqrt{x}\right| \leq \frac{|x_n - x|}{\sqrt{x}} < \frac{\epsilon\sqrt{x}}{\sqrt{x}} = \epsilon$$

□

**Theorem 2.1.7.** *If $x \in \mathbb{R}$ then there is a sequence $r_n$ from $\mathbb{Q}$ s.t. $r_n \to x$ as $x \to \infty$.*

*Proof.* For $n \in \mathbb{N}$ by density of $\mathbb{Q}$ there is an $r_n \in \mathbb{Q}$ with

$$x - \frac{1}{n} < r_n < x + \frac{1}{n} \iff 0 \leq |r_n - x| < \frac{1}{n}$$

By squeeze theorem,

$$|r_n - x| \to 0 \text{ as } n \to \infty \iff r_n \to x \text{ as } n \to \infty$$

□

## 2.2 Increasing and decreasing

**Definition 2.2.1** (Increasing)**.** *$\{x_n\}$ is increasing means $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$. If $x_n < x_{n+1}$ then strictly increasing.*

**Definition 2.2.2** (Decreasing)**.** *$\{x_n\}$ is decreasing means $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$. If $x_n > x_{n+1}$ then strictly decreasing.*

**Theorem 2.2.1** (Monotone Convergence Theorem)**.** *If $x_n$ is increasing and bounded above, then $x_n \to \sup\{x_1, x_2, \dots\}$ as $n \to \infty$.*

*Proof.* Given $\epsilon > 0$, there is an $x_n \in E = \{x_1, x_2, \dots\}$ so that

$$\sup E - \epsilon < x_n \leq \sup E < \sup E + \epsilon$$

for $n \geq N$, $x_N \leq x_n$, so

$$\sup E - \epsilon < x_N \leq x_n < \sup E + \epsilon \iff |x_n - \sup E| < \epsilon$$

□

**Ex 2.2.1.** *If $0 < |a| < 1$ then $a^n \to 0$ as $n \to \infty$*

*Proof.* Consider $|a^n - 0| = |a|^n$, to prove $|a|^n \to 0$ as $n \to \infty$. Here, $0 < |a| < 1$, $|a|^2 < |a|$. If $|a|^n < |a|^{n+1}$ then $|a|^{n+1} < |a|^{n+2}$. By induction, $|a|^{n+1} < |a|^n$ for all $n \in \mathbb{N}$. $|a|^n$ is decreasing and bounded below by $0$. By MCT, $|a|^n \to L$ as $n \to \infty$. Now note that

$$|a|^{2n} = |a|^n |a|^n \to L \cdot L = L^2 \text{ as } n \to \infty$$

But $|a|^{2n}$ is the subseq of every terms of $|a|^n$. By the subsequence theorem,

$$|a|^{2n} \to L \text{ as } n \to \infty$$

Therefore $L^2 = L$, gives that $L = 0, 1$. Since $|a| < 1$ and $|a|^n$ is decreasing, $L = 0$. □

**Ex 2.2.2.** *If $a > 0$ then $a^{1/n} \to 1$ as $n \to \infty$*

*Proof.* For $a > 1$ and $n < m$,

$$a^n < a^m$$

Take the $\frac{1}{mn}$ root, we get

$$\begin{aligned}
(a^n)^{\frac{1}{mn}} &< (a^m)^{\frac{1}{mn}} \\
a^{\frac{1}{m}} &< a^{\frac{1}{n}} \\
1 < a^{\frac{1}{n+1}} &< a^{\frac{1}{n}}
\end{aligned}$$

$a^{\frac{1}{n}}$ is decreasing and bounded below by 1. By the MCT, $a^n \to L$ as $n \to \infty$. But

$$a^{\frac{1}{2}n} = \sqrt{a^{\frac{1}{n}}} \to \sqrt{L}$$

Therefore $L = \sqrt{L}$, $L^2 = L$, $L = 1, 2$. Since it's bounded below by 1, $L = 1$. □

**Ex 2.2.3.** *If $0 < x_1 < 1$, $x_{n+1} = 1 - \sqrt{1 - x_n}$ for $n \in \mathbb{N}$, prove that $x_n$ converges and find the limit.*

*Proof.* Induct on $x_n$

*Base case.*

$$\begin{aligned}
x_2 &= 1 - \sqrt{1 - x_1} \\
&0 < x_1 < 1 \\
&0 > -x_1 > -1 \\
&1 > 1 - x_1 > 0 \\
&1 > \sqrt{1 - x_1} > 1 - x_1 > 0 \\
&-1 < -\sqrt{1 - x_1} < x_1 - 1 < 0 \\
&0 < 1 - \sqrt{1 - x_1} < x_1 < 1 \\
x_2 &< x_1 < 1
\end{aligned}$$

*Inductive step.* Suppose $0 < x_n < 1$ repeat the argument with $x_1$ replaced by $x_n$,

$$0 < x_{n+1} < x_n < 1$$

By induction the sequence is decreasing and is bounded below by 0. Therefore by MCT it converges. Now find the $L$. $x_n \to L$ as $n \to \infty$.

$$\begin{aligned}
L &= 1 - \sqrt{1 - L} \\
\sqrt{1 - L} &= 1 - L \\
1 - L &= 1, 0 \\
L &= 1, 0 \\
\Rightarrow L &= 0
\end{aligned}$$

□

**Ex 2.2.4.** *$x_0 > 0$ and $x_n = \frac{3}{2} + \frac{2}{3}x_{n-1}$ for $n \in \mathbb{N}$. Show $x_n$ converges and find the limit.*

*Proof.*

$$x_1 = \frac{2}{3} + \frac{2}{3}x_0 < \frac{1}{3}x_0 + \frac{2}{3}x_0 = x_0$$
$$2 < x_1 < x_0$$

If $2 < x_{n+1} < x_n$ then the same arg gives $2 < 2_{x+2} < x_{n+1}$. By induction, $x_n$ converses and bounded below by 2. By MCT, $x_n \to L$ as $n \to \infty$. Now find $L$.

$$L = \frac{2}{3} + \frac{2}{3}L$$
$$L = 2$$

□

**Ex 2.2.5.** *$x_0 < 3$, $x_n = \frac{3}{7} + \frac{6}{7}x_{n-1}$, prove it converges and find the limit.*

*Proof.*

$$x_0 = \frac{1}{7}x_0 + \frac{6}{7}x_0 < \frac{3}{7} + \frac{6}{7}x_0 = x_1 < \frac{3}{7} + \frac{18}{7} = 3$$
$$x_n = \frac{1}{7}x_n + \frac{6}{7}x_n < \frac{3}{7} + \frac{6}{7}x_n = x_{n+1} < \frac{3}{7} + \frac{18}{7} = 3$$

$x_n$ is increasing and bounded above. By MCT, $x_n \to L$ as $n \to \infty$. Taking the limit in

$$x_n = \frac{3}{7} + \frac{6}{7}x_{n-1}$$

We get

$$L = \frac{3}{7} + \frac{6}{7}L$$
$$L = 3$$

□

# Chapter 3

# Functions on $\mathbb{R}$

## 3.1 Limits

**Definition 3.1.1.** *Let $I$ be an open interval, $a \in I$, $\hat{I} = I \setminus \{a\}$, $f : \hat{I} \to \mathbb{R}$. $\lim_{x\to a} f(x) = L$ means*

$$\forall \epsilon > 0.\ \exists \delta > 0.\ \forall x \in \hat{I}.\ (0 < |x-a| < \delta \Rightarrow |f(x) - L| < \epsilon)$$

**Ex 3.1.1.** $f(x) = mx + b$, $a \in \mathbb{R}$

**Answer.** $\lim_{x\to a} f(x) = mx + b$

**Ex 3.1.2.** $f(x) = x^3 + x + 1$, *prove* $\lim_{x\to 2} f(x) = 11$

*Proof.* For $|x-2| < 1$, $1 < x < 3$ and $\left|x^2 + 2x + 5\right| \leq 20$. For $|x-2| < 1$ we have

$$\left|x^3 + x + 1 - 11\right| \leq \left|x^2 + 2x + 5\right| |x-2| \leq 20\,|x-2|$$

Given $\epsilon > 0$, let $\delta = \min(1, \frac{\epsilon}{20})$. For $0 < |x-2| < \delta$ we get

$$\left|x^3 + x + 1 - 11\right| \leq 20\,|x-2| < 20 \cdot \frac{\epsilon}{20} = \epsilon$$

$\square$

**Definition 3.1.2** (Sequential Characterization of Limits). *Let $x_n$ be sequence from $\hat{I}$,*

$$\lim_{x\to a} f(x) = L \iff \lim_{n\to\infty} x_n = a \to \lim_{n\Rightarrow\infty} f(x_n) = L$$

**Definition 3.1.3** (Polynomial). $n \in \mathbb{N}$, *a polynomial of defgree N*

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = \sum_{k=0}^{n} x_k x^k, \quad a_n \neq 0$$

**Theorem 3.1.1.**

$$\lim_{x \to x_0} P(x) = P(x_0)$$

*Proof.* Recall

$$\lim_{x \to x_0} mx + b = mx_0 + b$$

So,

$$\begin{aligned}
\lim_{x \to x_0} x^2 &= \lim_{x \to x_0} x \lim_{x \to x_0} x = x_0 \cdot x_0 = x_0^2 \\
\lim_{x \to x_0} x^3 &= \lim_{x \to x_0} x^2 \lim_{x \to x_0} x = x_0^2 \cdot x_0 = x_0^3
\end{aligned}$$

If $\lim_{x \to x_0} x^n = x_0^n$ then

$$\lim_{x \to x_0} x^{n+1} = \lim_{x \to x_0} x^n \lim_{x \to x_0} x = x_0^n \cdot x_0 = x_0^{n+1}$$

By induction $\lim_{x \to x_0} x^n = x_0^n$ for all $n \in \mathbb{N}$. For $P_n(x) = a_0 + a_1 x + \cdots + a_n x^n$,

$$\begin{aligned}
\lim_{x \to x_0} P_n(x) &= \lim_{x \to x_0} a_0 + \lim_{x \to x_0} a_1 x + \cdots + \lim_{x \to x_0} a_n x^n \\
&= a_0 + a_1 x_0 + \cdots + a_n x_0^n \\
&= P_n(x_0)
\end{aligned}$$

$\square$

**Definition 3.1.4** (Rational Function). *$P(x)$ and $Q(x)$ are polynomials with $Q(x) \neq 0$, then a rational function $R$ is*

$$R(x) = \frac{P(x)}{Q(x)} = \frac{\sum_{k=0}^{n} a_k x^k}{\sum_{k=0}^{m} b_k x^k}$$

**Theorem 3.1.2.**

$$\lim_{x \to x_0} R(x) = \frac{\lim_{x \to x_0} P(x)}{\lim_{x \to x_0} Q(x)} = \frac{P(x_0)}{Q(x_0)}, \quad Q(x_0) \neq 0$$

**Theorem 3.1.3.** *If $P(x)$ and $Q(x)$ are polynomials with $deg(P) \leq deg(Q)$. Then*

$$\lim \frac{P(x)}{Q(x)} = \begin{cases} 0 & \text{deg(P) < deg(Q)} \\ {}^{a_n}/_{b_n} & \text{deg(P) = deg(Q)} \end{cases}$$

*Proof.* For $m > n$

$$\begin{aligned}
&\lim_{x \to \infty} \frac{(a_n x^n + \cdots + a_0)}{(b_m x^m + \cdots + b_0)} \frac{\frac{1}{x_m}}{\frac{1}{x_m}} \\
=& \lim_{x \to \infty} \frac{(\frac{a_n x^n}{x^m} + \cdots + \frac{a_0}{x^m})}{(\frac{b_m x^m}{x^m} + \cdots + \frac{b_0}{x^m})} \\
=& \frac{0 + \cdots + 0}{b_m + 0 + \cdots + 0} = 0
\end{aligned}$$

$\square$

## 3.2 Convergence and Divergence

**Definition 3.2.1** (Convergence). $\lim_{x\to\infty} f(x) = L$ *means*

$$\forall\epsilon > 0.\ \exists N.\ x > N \Rightarrow |f(x) - L| < \epsilon$$

$\lim_{x\to-\infty} f(x) = L$ *means*

$$\forall\epsilon > 0.\ \exists N.\ x < N \Rightarrow |f(x) - L| < \epsilon$$

**Definition 3.2.2** (Divergence). $\lim_{x\to\infty} f(x) = \infty$ *means*

$$\forall M.\ \exists N.\ x > N \Rightarrow f(x) > M$$

$\lim_{x\to-\infty} f(x) = -\infty$ *means*

$$\forall M.\ \exists N.\ x < N \Rightarrow f(x) < M$$

**Definition 3.2.3** (One-sided limit). $\lim_{x\to a^+} f(x) = \infty$ *means,*

$$\forall M.\ \exists\delta > 0.\ a < x < a + \delta \Rightarrow f(x) > M$$

$\lim_{x\to a^-} f(x) = -\infty$ *means,*

$$\forall M.\ \exists\delta > 0.\ a - \delta < x < a \Rightarrow f(x) < M$$

**Ex 3.2.1.** *Prove*

$$\lim_{x\to-\infty} \frac{2x^2 + x + 5}{3x} = -\infty$$

*Proof.* Consider

$$\frac{2x^2 + x + 5}{3x} < \frac{1}{3}x$$

we want

$$\frac{1}{3}x < -(|M| + 1)$$

$$x < -3(|M| + 1)$$

Given $M$. Let $N = -3(|M| + 1)$. For $x < N$ we get

$$\frac{2x^2 + x + 5}{4x} < \frac{1}{3}x < \frac{1}{3}N = \frac{1}{3}(-3)(|M| + 1) = -(|M| + 1) < -|M| \leq M$$

□

**Ex 3.2.2.** *To prove*

$$\lim_{x\to1^+} \frac{2x}{x^3 - 1} = +\infty$$

*Proof.* Consider

$$\frac{2x}{x^3 - 1} = \frac{2x}{(x-1)(x^2 + x + 1)} = \frac{1}{x-1}\cdot\frac{2x}{x^2 + x + 1} > \frac{1}{x-1}\cdot\frac{2}{7} \quad \text{for } 1 < x < 2$$

We want

$$\frac{1}{x-1}\cdot\frac{2}{7} > |M| + 1$$

Given $M$, let $\delta = min(1, \frac{2}{7(|M|+1)})$ for $0 < x - 1 < \delta$. We get

$$\frac{2\delta}{x^3 - 1} > \frac{1}{x-1} \cdot \frac{2}{7} > \frac{7(|M|+1)}{2} \cdot \frac{2}{7} = |M| + 1 > |M| \geq M$$

□

## 3.3 Continuity

**Definition 3.3.1** (Continuity). $E \subseteq \mathbb{R}$, $f : E \to \mathbb{R}$, $a \in E$.

$$\lim_{x \to a} f(x) = f(a) \iff \forall \epsilon > 0.\ \exists \delta > 0.\ \forall x \in E.\ |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$$

- *$f$ is continuous at $a$*
- *If $f$ is continuous at every point $a \in E$, then $f$ is said to be continuous on $E$*

**Theorem 3.3.1** (Extreme Value Theorem). *$f$ is continuous on $[a, b]$, then*

$$\exists x_m, x_M \in [a, b].\ \forall x \in [a, b].\ f(x_m) \leq f(x) \leq f(x_M)$$

*Proof.* $f([a, b])$ is a nonempty set, then $\sup f([a, b])$ and $\inf f([a, b])$ exists. By definition of sup, there exists $y_n \in f([a, b])$ with $y_n \to \sup f([a, b])$ as $n \to \infty$, therefore there are $x_n \in [a, b]$ with $f(x_n) = y_n$. By B-W there is $x_{n_k} \to x_M \in [a, b]$. By continuity, $f(x_{n_k}) \to f(x_M)$, $y_{n_k} \to \sup f([a, b])$, gives that $f(x_M) = \sup f([a, b])$. Similarly, $f(x_m) = \inf f([a, b])$. □

**Theorem 3.3.2** (Intermediate Value Theorem). *If $f$ is cont on $[a, b]$ and if $y$ is any value between $f(a)$ and $f(b)$ then there is a $c$ between $a, b$ with $f(c) = y$.*

*Proof.* Suppose without lost of generality, $f(a) < y < f(b)$. SInce $f$ is cont at a, for $\epsilon = \frac{y - f(a)}{2} > 0$ there is a $\delta > 0$ so that $x \in [a, b]$, $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \frac{y - f(a)}{2}$, so $f([a, a + \delta])$ is bounded above by $y$. Let $E = \{t \mid f([a, t))\}$ is bounded above by $y$. Consider $\sup E$, exists by Completeness. $E$ is nonempty $t = a + \delta \in E$, bounded above by $b$. Let $t_n \in E$, $t_n \to \sup E$, gives that $f(t_n) \leq y$. By Continuity, $f(t_n) \to f(\sup E) \leq y$.

If $f(\sup E) = y$, done. If $f(\sup E) < y$, then by Continuity of $f$ at $\sup E$, for $\epsilon = \frac{y - f(\sup E)}{2}$ there is a $\hat{\delta} > 0$ such that

$$|x - a| < \hat{\delta} \Rightarrow \left| f(x) - f(\sup E) < \frac{y - \sup E}{2} \right|$$

$f(x) < \frac{y + \sup E}{2} < y$ for all $x \in (\sup E - \delta, \sup E + \delta)$. Contradiction. □

**Ex 3.3.1.** *If $f$ is continuous on $[a, b] = I$ then $f(I) = J$ is a closed bounded interval.*

*Proof.* By the extreme value theorem, $\exists x_m, x_M \in [a, b]$. so that $\forall x \in [a, b]$. $f(x_m) \leq f(x) \leq f(x_M)$. This shows that

$$f([a, b]) \leq [f(x_m), f(x_M)]$$

Let $f(x_m) < y < f(x_M)$. By the IVT there is an $x$ between $x_m, x_M$ with $y = f(x)$. This shows

$$[f(x_m), f(x_M)] \subseteq f([a, b])$$

which means

$$J = [f(x_m), f(x_M)]$$

□

**Ex 3.3.2.** *$f$, $g$ are cont on $[a, b]$ with $f(a) < g(a)$ and $f(b) > g(b)$. Shows that $\exists c \in [a, b]$. $f(c) = g(c)$*

*Proof.* Consider $h(x) = f(x) - g(x)$. Then $h(a) < 0 < h(b)$. By IVT, there is a $c$ between $a, b$ with $h(c) = 0$. This shows that $f(c) = g(c)$. □

## 3.4 Uniform continuity

**Definition 3.4.1** (Uniform Continuity)**.** $f : E \to \mathbb{R}$ *is uniform continuous means*

$$\forall \epsilon > 0.\ \exists \delta > 0.\ \forall x, y \in E.\ |x - y| < \delta \Rightarrow [f(x) - f(y) < \epsilon]$$

**Theorem 3.4.1.** *If* $f : E \to \mathbb{R}$ *is unif cont,* $x_n$ *from* $E$ *is Cauchy, then* $f(x_n)$ *is Cauchy.*

**Ex 3.4.1.** $f(x) = \frac{1}{x}$ *is not unif cont on* $(0, 1)$

*Proof.* Let $x_n = \frac{1}{n+1}, n \in \mathbb{N}$,

$$\begin{gathered} x_n \in (0,1) \\ x_n \to 0 \text{ as n } n \to \infty \end{gathered}$$

therefore $x_n$ is Cauchy. If $f(x) = 1/x$ were unif cont then $f(x_n)$ is Cauchy, and therefore converges. But $f(x_n) = \frac{1}{1/n+1} = n + 1$ which diverges to $\infty$, contradiction, therefore $f(x) = 1/x$ is not unif cont. □

**Theorem 3.4.2.** $f$ *cont on* $[a, b]$*, f is uniformly continuous on* $[a, b]$

**Theorem 3.4.3.** $f$ *cont on* $(a, b)$*, f is uniformly continuous iff*

$$\left.\begin{array}{l} \lim_{x \to a^+} f(x) \\ \lim_{x \to b^-} f(x) \end{array}\right\} \text{exists}$$

*Proof.* (of $\Rightarrow$) Let $x_n \in (a, b)$ with $x_n \to a$ as $n \to \infty$. $x_n$ is Cauchy. Then $f(x_n)$ is Cauchy. Therefore there exists $L$ s.t. $f(x_n) \to L$ as $x \to \infty$. Let $y_n \in (a, b)$ with $y_n \to a$ as $n \to \infty$. Similarly, there exists $K$ s.t. $f(y_m) \to K$ as $m \to \infty$. Consider

$$\begin{aligned} |L - K| &= |L - f(x_n) + f(x_n) - f(y_m) + f(y_m) - K| \\ &= |L - f(x_n)| + |f(x_n) - f(y_m)| + |f(y_m) - K| \end{aligned}$$

Given $\epsilon > 0$,

$$\begin{aligned} \exists N_1.\ n \geq N_1 &\Rightarrow |L - f(x_n)| < \frac{\epsilon}{3} \\ \exists N_2.\ m \geq N_2 &\Rightarrow |f(y_m) - K| < \frac{\epsilon}{3} \\ \exists \delta > 0.\ |x - y| < \delta &\Rightarrow |f(x) - f(y)| < \frac{\epsilon}{3} \end{aligned}$$

For this $\delta$, there is an $N_3$ such that

$$n \geq N_3 \Rightarrow |x_n - a| < \frac{1}{2}\delta$$

And there is an $N_4$ such that

$$m \geq N_4 \Rightarrow |y_m - a| < \frac{1}{2}\delta$$

Then for $n, m \geq \max(N_3, N_4)$

$$\begin{aligned} |x_n - y_m| &= |x_n - a + a - y_m| \\ &\leq |x_n - a| + |a - y_m| \\ &< \frac{1}{2}\delta + \frac{1}{2}\delta = \delta \end{aligned}$$

This gives

$$\forall \epsilon > 0.\ |L - K| < \epsilon$$
$$L = K$$

So for any sequence $x_n \in (a, b)$ with $x_n \to a$, $f(x_n) \to L$. By SCL

$$\lim_{x \to a^+} f(x) = L$$

Similarly,

$$\lim_{x \to b^-} f(x) = K$$

□

**Theorem 3.4.4.** *If $f(x)$ satisfies on $E$*

$$|f(x) - f(y)| \leq C\,|x - y|$$

*Then $f$ is unif cont on $E$.*

*Proof.* Given $\epsilon > 0$ let $\delta = {}^{\epsilon}/_{C} > 0$ and for $|x - y| < \delta$ we get

$$|f(x) - f(y)| \leq C\,|x - y| < C \cdot \frac{\epsilon}{C} = \epsilon$$

□

**Theorem 3.4.5.** *$g(x) = |x|$ is uniformly continuous*

*Proof.* Consider

$$||x| - |y|| \leq |x - y|$$

Given $\epsilon > 0$, let $\delta = \epsilon$ for $|x - y| < \delta$ we get $||x| - |y|| \leq |x - y| < \delta = \epsilon$ □

**Ex 3.4.2.** *$f(x) = x\log({}^{1}/_{x})$ on $(0, 1)$. Is it unif cont on $(0, 1)$?*

**Answer.**

$$\begin{aligned}
\lim_{x \to 0^+} x \log\left(\frac{1}{x}\right) &= \lim_{x \to 0^+} \frac{\log(\frac{1}{x})}{{}^{1}/_{x}} \\
&= \lim_{x \to 0^+} \frac{-\frac{x^{-2}}{x^{-1}}}{-x^{-2}} \\
&= \lim_{x \to 0^+} x \\
&= 0 \\
\lim_{x \to 1^-} x \log\left(\frac{1}{x}\right) &= 0
\end{aligned}$$

It is uniformly continuous.

**Ex 3.4.3.** *Use the definition to prove that $f(x) = 3x^2 + x + 5$ is unif cont on $[1, 4]$*

*Proof.* Consider

$$\begin{aligned}
|f(x) - f(y)| &= \left|3x^2 + x + 5 - (3y^2 + y + 5)\right| \\
&= \left|3(x^2 - y^2) + x - y\right| \\
&= |3(x + y)(x - y) + x - y| \\
&= |x - y|\,|3(x + y) + 1| \\
&\leq 25\,|x - y|
\end{aligned}$$

Given $\epsilon > 0$, let $\delta = \frac{\epsilon}{25}$. For $|x-y| < \delta$, we get

$$|f(x) - f(y)| \le 25\,|x-y| < 25 \cdot \frac{\epsilon}{25} = \epsilon$$

□

**Ex 3.4.4.** *$f(x) = x$ is unif cont on $(1,0)$ since $|f(x) - f(y)| = |x-y|$. Given $\epsilon > 0$, let $\delta = \epsilon$, then $|x-y| < \delta \Rightarrow |f(x) - f(y)| = |x-y| < \epsilon$*

**Theorem 3.4.6.** *$f, g : E \to \mathbb{R}$ are unif cont and bounded, $f \cdot g : E \to \mathbb{R}$ is unif cont.*

*Proof.* Since $f, g$ are bounded, $\exists M, K > 0.$ s.t.

$$\begin{aligned} \forall x \in E.\ |f(x)| \le M \\ \forall x \in E.\ |g(x)| \le K \end{aligned}$$

Consider

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &= |f(x)g(x) - f(y)g(x) + f(y)g(x) - f(y)g(y)| \\ &\le |g(x)|\,|f(x) - f(y)| + |f(y)|\,|g(x) - g(y)| \\ &\le K\,|f(x) - f(y)| + M\,|g(x) - g(y)| \end{aligned}$$

Given $\epsilon > 0$ there is $\delta_1, \delta_2$ s.t.

$$\begin{aligned} \forall x, y \in E.\ |x-y| < \delta_1 \Rightarrow |f(x) - f(y)| < \frac{\epsilon}{2K} \\ \forall x, y \in E.\ |x-y| < \delta_2\ \Rightarrow |g(x) - g(y)| < \frac{\epsilon}{2M} \end{aligned}$$

Let $\delta = \min(\delta_1, \delta_2)$. If $|x-y| < \delta$, we get

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &\le K\,|f(x) - f(y)| + M\,|g(x) - g(y)| \\ &< K \cdot \frac{\epsilon}{2K} + M \cdot \frac{\epsilon}{2M} \\ &= \epsilon \end{aligned}$$

□

# Chapter 4

# Differentiability on $\mathbb{R}$

## 4.1 Derivative

**Definition 4.1.1** (Derivative). *$I$ is an open interval of $\mathbb{R}$, $a \in I$, $f : I \to \mathbb{R}$. $f$ is differentiable at $a$ means*

$$\exists f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

## 4.2 Differentiability

**Definition 4.2.1** (Differentiability). *If $f$ is differentiable at every point of $I$, then $f$ is differentiable on $I$. We have $f' : I \to \mathbb{R}$. If $I'$ is continuous on $I$ then we say that $f \in C^1(I)$.*

**Definition 4.2.2.** *$o(h)$ means $\lim_{h \to 0} \frac{o(h)}{h} = 0$, and it goes to zero faster than $h$ as $h \to 0$.*

**Theorem 4.2.1.** *$h$ is not $o(h)$*

**Theorem 4.2.2.** *If $f$ is differentiable at $a$, then*

$$f(a+h) - f(a) - f'(a)h = o(h), \quad \text{for } |h| < \delta$$

*Proof.*

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - f'(a)h}{h} = \lim_{h \to 0} \left( \frac{f(a+h) - f(a)}{h} - f'(a) \right) = 0$$

□

**Remark 4.2.1.** $f(a+h) = f(a) + f'(a)h + o(h)$

**Theorem 4.2.3.** *If $\exists m.\ f(a+h) - f(a) - mh = o(h)$, $f$ is differentiable at $a$ and $f'(a) = m$*

*Proof.*

$$\begin{aligned} \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h \to 0} \frac{f(a+h) - f(a) - mh + mh}{h} \\ &= \lim_{h \to 0} \left( \frac{o(h)}{h} + m \right) \\ &= m \end{aligned}$$

□

**Theorem 4.2.4.** *$f$ differentiable at $a$ implies $f$ is continuous at $a$*

*Proof.* Note that

$$\lim_{x\to a} f(x) = f(a) \iff \lim_{x\to a}(f(x) - f(a)) = 0$$

Consider

$$\begin{aligned}\lim_{x\to a}(f(x) - f(a)) &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \cdot (x - a) \\ &= f'(a) \cdot 0 = 0\end{aligned}$$

$\square$

**Remark 4.2.2.** *If $f$ is not continuous at $a$, then $f$ is not differentiable at $a$*

**Ex 4.2.1.** *$f(x) = |x|$ is not differentiable at $x = 0$.*

*Proof.* $\lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|}{h}$, does not exist. $\square$

**Ex 4.2.2.** *$f$ is not differentiable at $x = 0$.*

$$f(x) = \begin{cases} x\sin(1/x) & x \neq 0 \\ 0 & x \neq 0 \end{cases}$$

**Ex 4.2.3.**

$$f(x) = \begin{cases} x^2\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

**Answer.** At $a = 0$, $\lim_{h\to 0} \frac{h^2\sin(1/h)}{h} = \lim_{h\to 0} h\sin(1/h) = 0$. But $f \notin C^1(\mathbb{R})$.

**Theorem 4.2.5.** *$f, g$ differentiable at $a$ implies $f + g, fg$ are differentiable at $a$. Moreover*

$$(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a)$$

**Ex 4.2.4.** *True or False. $f = g^2$, $f$ is differentiable on $[a, b]$ implies $g$ is differentiable on $(a, b)$.*

**Answer.** False. Let

$$g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \notin \mathbb{Q} \end{cases}$$

$g$ is nowhere continuos so nowhere differentiable. But $f(x) = 1$ and differentiable on $[a, b]$.

**Ex 4.2.5.** *True or False. $f$ is differentiable on $(a, b]$ and*

$$\frac{f(x)}{x - a} \to 1 \text{ as } x \to a^+$$

*then $f$ is uniformly continuous on $(a, b)$*

**Answer.** True. $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} \frac{f(x)}{x-a}(x - a) = 1 \cdot 0 = 0$. Since $\lim_{x\to b^-} f(x) = f(b)$. By uniformly continuous theorem, $f$ is uniformly continuous on $[a, b]$.

**Theorem 4.2.6.** *For* $f(x) = x^n$*,* $n \in \mathbb{N}$*.* $f'(x) = nx^{n-1}$

*Proof.*

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \\
&= \lim_{x\to a} \frac{x^n - a^n}{x - a} \\
&= \lim_{x\to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})}{x - a} \\
&= \lim_{x\to a} (x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1}) \\
&= a^{n-1} + a^{n-2} \cdot a + \cdots + a \cdot a^{n-2} + a^{n-1} \\
&= na^{n-1} \\
f'(x) &= nx^{n-1}
\end{aligned}$$

□

**Theorem 4.2.7.** $f(x) = \frac{1}{x^n} = x^{-n}$*,* $n \in \mathbb{N}$*,* $x \neq 0$*.* $f'(x) = -nx^{-n-1}$

*Proof.*

$$\begin{aligned}
f'(a) &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \\
&= \lim_{x\to a} \frac{\frac{1}{x^n} - \frac{1}{a^n}}{x - a} \\
&= \lim_{x\to a} \frac{\frac{-(x^n - a^n)}{a^n x^n}}{x - a} \\
&= \lim_{x\to a} \frac{-(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})}{a^n x^n} \\
&= \frac{-na^{n-1}}{a^{2n}} \\
&= -na^{-n-1} \\
f'(x) &= -nx^{-n-1}
\end{aligned}$$

□

**Theorem 4.2.8.** $f : (0, \infty) \to \mathbb{R}$*,* $f(x) - f(y) = f\left(\frac{x}{y}\right)$*,* $f(1) = 0$*. Then*

*(a) If* $f$ *is continuous at* $1$*,* $f$ *is continuous on* $(0, \infty)$*.*

*Proof.* Since $f$ is continuous at 1,

$$\forall \epsilon > 0.\ \exists \delta > 0.\ (|x - 1| < \delta \Rightarrow |f(x) - f(1)| < \epsilon)$$

Let $a \in (0, \infty)$, consider

$$|f(x) - f(a)| = \left|f\left(\frac{x}{a}\right)\right| < \epsilon \text{ when } \left|\frac{x}{a} - 1\right| < \delta_1$$

Given $\epsilon > 0$, let $\delta = a\delta_1$. If $|x - a| < \delta$ then $f(x) - f(a) < \epsilon$ □

(b) *If $f$ is differentiable at $1$, $f$ is differentiable on $(0, \infty)$.*

*Proof.* Since $f$ is differentiable at 1

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to a} \frac{f(x)}{x - 1}$$

Let $a \in (0, \infty)$, consider

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{f\left(\frac{x}{a}\right)}{x - a} = \lim_{x \to a} \frac{f\left(\frac{x}{a}\right)}{a\left(\frac{x}{a} - 1\right)} = \frac{1}{a} \lim_{x \to a} \frac{f\left(\frac{x}{a}\right)}{\frac{x}{a} - 1} = \frac{1}{a} f'(1)$$

□

**Definition 4.2.3** (Local Maximum and Minimum)**.** *$f : I \to \mathbb{R}$, $I$ is an open interval. $f$ has a local maximum at $c$ means*

$$\exists \delta > 0.\ |x - c| < \delta \Rightarrow f(x) \leq f(c)$$

*$f$ has a local minimum at $c$ means*

$$\exists \delta > 0.\ |x - c| < \delta \Rightarrow f(x) \geq f(c)$$

**Theorem 4.2.9.** *If $f$ has an local maximum at $c$ and is differentiable at $c$, then $f'(c) = 0$.*

*Proof.*

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}$$

$$f(c+h) - f(c) \leq 0$$

□

**Definition 4.2.4** (Even and Odd Function)**.** *$f : (-a, a) \to \mathbb{R}$. $f$ is **even** means $f(x) = f(-x)$. $f$ is **odd** means $f(-x) = -f(x)$*

**Theorem 4.2.10.** *$f$ is differentiable on $(-a, a)$. $f$ is odd implies $f'$ is even.*

*Proof.*

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
f'(-x) &= \lim_{h \to 0} \frac{f(-x+h) - f(-x)}{h} \\
&= \lim_{h \to 0} \frac{-f(x-h) + f(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x-h) - f(x)}{-h} \\
&= \lim_{-h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= f'(x)
\end{aligned}$$

□

**Theorem 4.2.11** (Chain Rule)**.** $f : A \to \mathbb{R}$, $g : B \to \mathbb{R}$, $f(A) \subseteq B$. *$f$ is differentiable at $a \in A$, $g$ is differentiable at $b \in f(a)$. Then $g \circ f$ is differentiable at $a$ and $(g \circ f)'(a) = g'(f(a))f'(a)$*

*Proof.* By definition,

$$\begin{aligned} f(a+h) &= f(a) + f'(a)h + o(h) \\ g(b+k) &= g(b) + g'(b)k + o(k) \\ (g \circ f)(a+h) &= g(f(a+h)) \\ &= g(f(a) + f'(a)h + o(h)) \end{aligned}$$

Let $k = f'(a)h + o(h)$

$$\begin{aligned} &= g(b+k) \\ &= g(b) + g'(b)k + o(k) \\ &= g(b) + g'(b)f'(a)h + g'(b)o(h) + o(h) \\ &= (g \circ f)(a) + (g \circ f)'(a)h + o(h) \end{aligned}$$

□

## 4.3 The Mean Value Theorem

**Theorem 4.3.1** (Mean Value Theorem)**.** *$f : [a, b] \to \mathbb{R}$. $f$ is continuous on $[a, b]$, differentiable on $(a, b)$. Then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$*

**Theorem 4.3.2.** *If $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$ and $f'(x) > 0$ for all $x \in (a, b)$ then $f$ is strictly increasing.*

*Proof.* For $x, y \in (a, b)$ with $x < y$, apply MVT to $f$ on $[x, y] \subseteq [a, b]$. Then

$$\exists c \in (x, y).\ f(y) - f(x) = f'(c)(y - x) > 0$$

□

**Theorem 4.3.3.** *If $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$ and $f'(x) = 0$ for all $x \in (a, b)$ then $f$ is a constant function.*

*Proof.* For $x, y \in (a, b)$ with $x < y$, apply MVT to $f$ on $[x, y] \subseteq [a, b]$. Then

$$\exists c \in (x, y).\ f(y) - f(x) = f'(c)(y - x) = 0$$

□

**Corollary 4.3.1.** *$f, g$ continuous on $[a, b]$, differentiable on $(a, b)$ and $f'(x) = g'(x)$ for all $x \in (a, b)$, then $\forall x \in [a, b].\ g(x) = f(x) + c$.*

*Proof.* Let $h(x) = g(x) - f(x)$. $h$ is continuous on $[a, b]$, differentiable on $(a, b)$ and $h'(x) = g'(x) - f'(x) = 0$ for all $x \in (a, b)$. By the previous theorem, $h$ is a constant function. Then $g(x) = f(x) + c$ for some $c \in \mathbb{R}$. □

**Theorem 4.3.4** (Generalized Mean Value Theorem)**.** *If $f, g$ are continuos on $[a, b]$ and differentiable on $(a, b)$ then $\exists c \in (a, b)$. with*

$$(f(b) - f(a)) \cdot g'(c) = (g(b) - g(a)) \cdot f'(c)$$

*Proof.* Let

$$h(x) = (f(b) - f(a))(g(x) - g(a)) - (g(b) - g(a))(f(x) - f(a))$$

$h$ is continuous on $[a, b]$, differentiable on $(a, b)$. $h(a) = h(b) = 0$.

$$h'(x) = (f(b) - f(a)) \cdot g'(c) - (g(b) - g(a)) \cdot f'(c)$$

Apply Rolle's Theorem to $h$ on $[a, b]$. Then $h'(c) = 0$ for some $c \in (a, b)$. □

**Theorem 4.3.5.** *If $f$ is increasing on $[a,b]$, then for all $c \in [a,b)$*

$$f(c+) = \lim_{x \to c+} f(x) \quad \textit{exists}$$

*For all $c \in (a,b]$*

$$f(c-) = \lim_{x \to c+} f(x) \quad \textit{exists}$$

*When $f(c+) \neq f(c-)$, then $f$ is not continuous at $c$. A jump discontinuity $j(c) = f(c+) - f(c-)$.*

*Proof.* Let $c \in [a,b)$. Consider the set $f((c,b))$, since $f$ is increasing , this set is bounded below by $f(c)$. Call $f(c+) = \inf f((c,b))$. For any $\epsilon = \frac{1}{n}$ there is a point $f(x_n) \in f((c,b))$, then

$$f(c+) \leq f(x_n) < f(c+) + \frac{1}{n}$$

by Squeeze Theorem, $f(x_n) \to f(c+)$ as $n \to \infty$. Given $\epsilon > 0$ there exists $N$ so that

$$n \geq N \Rightarrow |f(x_n) - f(c+) < \epsilon|$$

Since $x_n \in (c,b)$, $c < x_n$, $f(c+) \leq f(x_n)$, $0 \leq f(x_n) - f(c+) < \epsilon$. For $c < x < x_n$, $f(x) \leq f(x_n)$ since $f$ is increasing. Then

$$0 \leq f(x) - f(c+) \leq f(x_n) - f(c+) < \epsilon$$

Let $\delta = x_n - c$, for $c < x < c + \delta$ we have $|f(x) - f(c+)| < \epsilon$. $\square$

**Theorem 4.3.6.** *If $f$ is monotone on $[a,b]$ that $f$ has at most a countable set of jump discontinuity.*

*Proof.* For increasing function $f$, recall at a jump discontinuity, $c \in (a,b)$, $j(c) = f(c+) - f(c-)$. If there are discontinuity at $c$ and $\cap c \in (a,b)$ with $c < \cap c$, $f(c+) \leq f(\cap c-)$. For $K \in \mathbb{N}$ let $E_k = \left\{c \in (a,b) \mid j(c) < \frac{1}{k}\right\}$. If there are N then

$$\begin{aligned} N \cdot \frac{1}{k} &\leq \sum_{i=1}^{\infty} j(c_i) \leq f(b) - f(a) \\ &\Rightarrow N \leq k \cdot (f(b) - f(a)) \\ &\Rightarrow E_k \text{ is finite} \end{aligned}$$

then $E = \bigcup_{k=1}^{\infty} E_k$ is all jump discontinuity. $E$ is at most countable. $\square$

**Ex 4.3.1.** *Prove that $1 + x < e^x$ for $x > 0$*

*Proof.* Let $f(x) = e^2 - (1+x)$, $f(0) = 0$, $f'(x) = e^x - 1$. $f'$ is continuous and differentiable on $[0,\infty)$. $f''(x) = e^x > 0$ for all $x \in \mathbb{R}$. $g$ is strictly increasing, for $0 < x$, $g(0) < g(x)$, $0 < e^x - 1$, then $f$ is strictly increasing. For $0 < x$, $f(0) < f(x)$, $e^x - (1+x) > 0$, $e^x > 1 + x$. $\square$

**Ex 4.3.2.** *Prove that $(1+x)^x \leq 1 + \alpha x$, for $0 < \alpha \leq 1$ and $x \geq 0$.*

*Proof.* When $\alpha = 1$, $1 + x \leq 1 + x$. Set $f(x) = (1+x)^\alpha$ is continuous and differentiable for $0 < \alpha \leq 1$, $x \geq 0$.

$$f'(x) = \alpha(1+x)^{\alpha-1}$$

For $x > 0$, apply MVT for $f$ on $[0,x]$.

$$\begin{aligned} \exists c \in (0,x).\ f(x) - f(0) &= f'(c) \cdot x \\ (1+x)^\alpha - 1 &= \alpha(a+c)^{\alpha-1} \cdot x < \alpha \cdot x \\ (1+x)^\alpha &< \alpha \cdot x \end{aligned}$$

$\square$

## 4.4 Taylor's Theorem and L'Hospital's Rule

**Observe** If $f$ is differentiable on $(a, b)$ and $x, x_0$ are two points from $(a, b)$, apply MVT to $f$. From $x$ to $x_0$, then $\exists c \in [x, x_0]$. $f(x) - f(x_0) = f'(c)(x - x_0)$, or $f(x) = f'(c)(x - x_0) + f(x_0)$.

**Definition 4.4.1** (Taylor's Polynomial). *If $f$ is $n \in \mathbb{N}$ times differentiable on $(a, b)$ then*

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

*is the Taylor's Polynomial of degree $n$.*

**Theorem 4.4.1** (Taylor's). *If $f$ is $n + 1$ times differentiable on $(a, b)$ then*

$$f(x) = P_n(x) + \underbrace{\frac{f^{n+1}(c)}{(n+1)!}(x - a)^{n+1}}_{R_n(x)}$$

*where $R_n(x)$ is the error in approximating $f(x)$ by $P_n(x)$.*

*Proof.* Let

$$G(t) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(t)(x - t)^k}{k!}$$

is continuos and differentiable on the interval from $x$ to $x_0$. Then

$$G(x) = 0,\ G(x_0) = f(x) - P_n(x),\ G'(t) = -\frac{f^{(n+1)}(t)}{n!}(x - t)^n$$

Now let

$$H(t) = \frac{(x - t)^{n+1}}{(n+1)!}$$

is continuos and differentiable on the interval from $x$ to $x_0$. Then

$$H(x) = 0,\ H(x_0) = \frac{(x - x_0)^{n+1}}{(n+1)!},\ H'(t) = -\frac{(x - t)^n}{n!}$$

Apply the GMVT to $G, H$ on the interval from $x$ to $x_0$. Then there exists $c \in (x, x_0)$ such that

$$\begin{aligned}
(G(x) - G(x_0))H'(c) &= (H(x) - H(x_0))G'(c) \\
(f(x) - P_n(x))\frac{(x - c)^n}{n!} &= \frac{(x - x_0)^{n+1}}{(n+1)!} \cdot \frac{f^{(n+1)}(c)}{n!}(x - c)^n \\
f(x) - P_n(x) &= \frac{(x - x_0)^{n+1}}{(n+1)!} \cdot f^{(n+1)}(c) \\
f(x) &= \frac{(x - x_0)^{n+1}}{(n+1)!} \cdot f^{(n+1)}(c) + P_n(x)
\end{aligned}$$

$\square$

**Ex 4.4.1.** $f(x) = e^x,\ x_0 = 0$

**Answer.**

$$\begin{aligned} P_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k \\ &= \sum_{k=0}^{n} \frac{x^k}{k!} \end{aligned}$$

For some $c \in (0, x)$

$$R_n(x) = \frac{f^{(n+1)}}{(n+1)!} x^{n+1} = \frac{e^c x^{n+1}}{(n+1)!}$$

If $-1 \leq x \leq 1$, then

$$|R_n(x)| \leq \frac{e}{(n+1)!} < .000009$$

**Ex 4.4.2.** $f(x) = \sin x,\ x_0 = 0$

**Answer.**

$$\begin{aligned} P_{2n+1}(x) &= \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \\ R_{2n+1}(x) &= \frac{f^{(2n+2)}(c)}{(2n+2)!} x^{2n+2} \qquad \text{(for some } c \in (0, x)) \end{aligned}$$

For $-1 \leq x \leq 1$, when $n = 4$

$$|R_9(x)| < .00000003$$

**Ex 4.4.3.** *What is*

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots - \frac{x^{4n-1}}{(4n-1)!}$$

**Answer.**

$$\sin(x) = P_{4n-1}(x) + R_{4n-1}(x)$$

**Ex 4.4.4.** $f(x) = \log x,\ x = 1$

**Answer.**

$$\begin{aligned} f^{(k)}(1) &= (-1)^{k+1}(k-1)! \\ P_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!} (x-1)^k \\ &= \sum_{k=1}^{n} \frac{f^{(k)}(1)}{k!} (x-1)^k \\ &= \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} (x-1)^k \\ R_n(x) &= f^{n+1}(c)(n+1)!(n-1)^{n+1} \\ &= \frac{\frac{(-1)^{n+2} n!}{(n+1)c^{n+1}}}{(n+1)!} (x-1)^{n+1} \\ &= \frac{(-1)^{n+2}}{(n+1)^2 c^{n+1}} (x-1)^{n+1} \end{aligned}$$

**Theorem 4.4.2** (L'Hospital's Rule)**.** *$I$ is an open interval, $a$ is in $I$ and is an endpoint of $I$. $f, g$ are differentiable on $I$ with $f(x) \neq 0 \neq g'(x)$ for all $x \in I$.*

$$\lim_{\substack{x \to a \\ x \in I}} f(x) = \lim_{\substack{x \to a \\ x \in I}} g(x) = 0 \text{ or } \infty \quad \Longrightarrow \quad \lim_{\substack{x \to a \\ x \in I}} \frac{f'(x)}{g'(x)} = B \in \bar{\mathbb{R}} = \lim_{\substack{x \to a \\ x \in I}} \frac{f(x)}{g(x)}$$

*Proof.*

(Easy case) $a \in \mathbb{R}$

$$\lim_{\substack{x \to a \\ x \in I}} f(x) = \lim_{\substack{x \to a \\ x \in I}} g(x) = 0$$

Define $f(a) = g(a) = 0$. Then for any $x \in I$, $f, g$ are continuos and differentiable on the interval from $a$ to $x$. Apply GMVT to $f, g$ on the interval from $a$ to $x$, then there exists $c \in (a, x)$ such that

$$(f(x) - f(a))g'(c) = (g(x) - g(a))f'(c)$$
$$\frac{f(x)}{g(x)} = \frac{f'(c_x)}{g'(c_x)}$$
$$\lim_{\substack{x \to a \\ x \in I}} \frac{f(x)}{g(x)} = \lim_{\substack{x \to a \\ x \in I}} \frac{f'(c_x)}{g'(c_x)} = \lim_{\substack{c_x \to a \\ x \in I}} \frac{f'(c_x)}{g'(c_x)} = B$$

(Harder case) $a = +\infty$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = 0$$

Given $\epsilon > 0$, $\exists N > 0$ so that

$$\begin{aligned} x > N &\Rightarrow |f(x) - 0| < \epsilon \\ &\Rightarrow |g(x) - 0| < \epsilon \end{aligned}$$

So for $y = \frac{1}{x}$

$$\begin{aligned} y < \frac{1}{N} \Rightarrow &\Rightarrow \left| f\left(\frac{1}{y}\right) - 0 \right| < \epsilon \\ &\Rightarrow \left| g\left(\frac{1}{y}\right) - 0 \right| < \epsilon \end{aligned}$$

$$\lim_{y \to 0^+} g(1/y) = 0 = \lim_{y \to 0^+} g(1/y)$$
$$\lim_{y \to 0^+} \frac{f'(1/y) \cdot \frac{1}{-y^2}}{g'(1/y) \cdot \frac{1}{-y^2}} = \lim_{y \to 0^+} \frac{f'(1/y)}{g'(1/y)} = B$$

$\square$