Advanced Calculus
Lecture Notes

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Chapter 1

1.1 Euclidean Space $\mathbb{R}^n$

$$
\begin{aligned}
\vec{x} \in \mathbb{R}^n \quad \vec{x} &= (\vec{x}_1, \vec{x}_2, \vec{x}_3, \dots, \vec{x}_n) \\
\vec{e}_1 &= (1, 0, 0, \dots, 0) \\
\vec{e}_2 &= (0, 1, 0, \dots, 0) \\
\vec{e}_3 &= (0, 0, 1, \dots, 0) \\
&\vdots \\
\vec{e}_n &= (0, 0, 0 \dots, 1) \\
\vec{a} + \vec{b} &= (\vec{a_1} + \vec{b_1}, \vec{a_2} + \vec{b_2}, \dots, \vec{a_n} + \vec{b_n}) && \text{(Addition)} \\
\alpha\vec{a} &= (\alpha\vec{a_1}, \alpha\vec{a_2}, \dots, \alpha\vec{a_n}) && \text{(Scalar Multiplication)} \\
\vec{a} \cdot \vec{b} &= \vec{a_1}\vec{b_1} + \vec{a_2}\vec{b_2} + \cdots + \vec{a_n}\vec{b_n} && \text{(Dot Product)} \\
|\vec{a}| &= \sqrt{\vec{a_1}^2 + \vec{a_2}^2 + \cdots + \vec{a_n}^2} && \text{(Norm)} \\
&= \sqrt{\vec{a} \cdot \vec{a}} \\
|\vec{e_j}| &= \sqrt{1} = 1 \\
\vec{e_i} \cdot \vec{e_j} &= \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} = \delta_{ij} && \text{(Kroneker delta)} \\
[\vec{e_i} \cdot \vec{e_j}] &= \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ & & \ddots & \end{bmatrix} = I_{n \times n} \\
f(t) &= \left|\vec{a} - t\vec{b}\right|^2 \geq 0 \\
&= (\vec{a} - t\vec{b}) \cdot (\vec{a} - t\vec{b}) \\
&= |\vec{a}|^2 - t\vec{a} \cdot \vec{b} - t\vec{b} \cdot \vec{a} + t^2 \left|\vec{b}\right|^2 \\
&= |\vec{a}|^2 - 2t\vec{a} \cdot \vec{b} + t^2 \left|\vec{b}\right|^2
\end{aligned}
$$

When $\vec{b} \neq 0$, $f(t)$ is a parabola of t opens upward, which has minimum when first derivative is 0.

$$
\begin{aligned}
f'(t) &= -2\vec{a} \cdot \vec{b} + 2t \left|\vec{b}\right|^2 = 0 \\
t &= \frac{\vec{a} \cdot \vec{b}}{\left|\vec{b}\right|^2}
\end{aligned}
$$

$$
\begin{aligned}
f\left(\frac{\vec{a}\cdot\vec{b}}{\left|\vec{b}\right|^2}\right) &\geq 0\\
|\vec{a}|^2 - 2\frac{\vec{a}\cdot\vec{b}}{\left|\vec{b}\right|^2}\vec{a}\cdot\vec{b} + \left(\frac{\vec{a}\cdot\vec{b}}{\left|\vec{b}\right|^2}\right)^2\left|\vec{b}\right|^2 &\geq 0\\
|\vec{a}|^2 - \frac{(\vec{a}\cdot\vec{b})^2}{\left|\vec{b}\right|^2} &\geq 0\\
-1 \leq \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\left|\vec{b}\right|} &\leq 1\\
\vec{a}\cdot\vec{b} &= |\vec{a}|\left|\vec{b}\right|\cos\theta\\
\vec{a}\cdot\vec{b} &= 0 \text{ when } 0 = {}^{\pi}/_{2}
\end{aligned}
$$

Cross product

In $\mathbb{R}^3$,

$$
\begin{aligned}
\vec{a}\times\vec{b} &= \vec{e_1}(a_2b_3 - a_3b_2) + \vec{e_2}(a_3b_1 - a_1b_3) + \vec{e_3}(a_1b_2 - a_2b_1)\\
\vec{a}\cdot\vec{a}\times\vec{b} &= a_1\vec{e_1}(a_2b_3 - a_3b_2) + a_2\vec{e_2}(a_3b_1 - a_1b_3) + a_3\vec{e_3}(a_1b_2 - a_2b_1) = 0\\
\vec{a} &\perp \vec{a}\times\vec{b}\\
\vec{a}\cdot\vec{a}\times\vec{b} &= 0\\
\vec{b} &\perp \vec{a}\times\vec{b}\\
\left|\vec{a}\times\vec{b}\right| &= |\vec{a}|\left|\vec{b}\right|\sin\theta
\end{aligned}
$$

1.2 Subsets of $\mathbb{R}^n$

Definition 1.1 (Balls).

$$B(\vec{a}, r) = \{\vec{x} \mid |\vec{x} - \vec{a}| < r\}, \quad r > 0$$

Definition 1.2 (Interior Point S^{int}). *$\vec{x}$ is an interior point of S means $B(\vec{x}, r) \subseteq S$, denoted with S^{int}.*

Definition 1.3 (Boundry Point ∂S). *$\vec{x}$ is a boundary point of S means*

$$
\begin{aligned}
\forall r > 0.\ B(\vec{x}, r) \cap S &\neq \varnothing\\
B(\vec{x}, r) \cap S^c &\neq \varnothing
\end{aligned}
$$

Denoted with ∂S.

Remark 1.1. $\partial S = \partial S^c$

Definition 1.4 (Open Set). *S is open when it contains none of its boundary points. Every point of S is an interior point.*

Definition 1.5 (Closed Set). *S is closed when it contains all of its boundary points. S^c is open.*

Definition 1.6 (Closure $\bar{S}$)**.** $\bar{S} = S \cup \partial S$

Ex 1.1.

$$S = \underbrace{\left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\right\}}_{B((0,0),1)} \setminus \{(0, y) \mid y \leq 0\}$$

S is an open set.

$$\partial S = \left\{(x, y) \mid x^2 + y^2 = 1\right\} \cup \{(0, y) \mid -1 \leq y \leq 0\}$$
$$\bar{S} = \left\{(x, y) \mid x^2 + y^2 \leq 1\right\}$$
$$\bar{S}^{int} = B((0, 0), 1) = \left\{(x, y) \mid x^2 + y^2 < 1\right\}$$

1.3 Continuity

Definition 1.7 (Continuity)**.**

$$u \in \mathbb{R}^n, \vec{f} : u \in \mathbb{R}^n$$
$$\vec{x} \in u,\ \vec{x} = (x_1, \ldots, x_n)$$
$$\vec{f}(\vec{x}) = (f_1(\vec{x}), \ldots, f_m(\vec{x}))$$

for $\vec{a} \in u$, $\vec{f}$ is continuous at $\vec{a}$ means

$$\forall \epsilon > 0.\ \exists \delta > 0.\ \vec{f}(u \cap B(\vec{a}, \delta)) \in B(\vec{f}(\vec{a}), \epsilon)$$

Ex 1.2.

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Answer. $\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = 0 = f(0, 0)$

Proof. Consider

$$\left|\frac{x^2 y}{x^2 + y^2} - 0\right| = \frac{x^2 |y|}{x^2 + y^2} \leq \frac{(x^2 + y^2) |y|}{x^2 + y^2} = |y| = \sqrt{y^2} \leq \sqrt{x^2 + y^2}$$

Because $\sqrt{x^2 + y^2} \to 0$ as $(x, y) \to (0, 0)$, by squeeze theorem, $\left|\frac{x^2 y}{x^2+y^2} - 0\right| \to 0$, that is, $\frac{x^2 y}{x^2+y^2} \to 0$ as $(x, y) \to (0, 0)$ □

Ex 1.3.

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^4+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Answer. Approach $(0, 0)$ along $y = mx^2$, $\lim = \frac{m}{1+m^2}$. Limit DNE.

Theorem 1.1. *If $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$ is continuous on $\mathbb{R}^n$ and $u \subseteq \mathbb{R}^m$ is open then $f^{-1}(u) \subseteq \mathbb{R}^n$ is open.*

1.4 Sequence in $\mathbb{R}^n$

Definition 1.8 (Sequence). *A sequence in $\mathbb{R}^n$ is $\vec{x}_k \quad k = 1, 2, 3, \ldots$.*

Definition 1.9 (Limit). $\vec{x}_k \to \vec{L}$ *as* $k \to \inf$ *means*

$$\forall \epsilon > 0.\ \exists N.\ k \geq N \Rightarrow \left|\vec{x}_k - \vec{L}\right| < \epsilon$$

Theorem 1.2. *If $\vec{x} \in \bar{S} \subseteq \mathbb{R}^n$, then there is a sequence in S that converges to $\vec{x}$.*

Proof. $\bar{S} = S \cup \partial S$. If $\vec{x} \in S$ then take $\vec{x}_k = \vec{x}$ then $\vec{x}_k \to \vec{x}$ as $k \to \infty$. Otherwise $\vec{x} \notin S$ and $\vec{x} \in \partial S$, then for $k = 1, 2, 3, \ldots$, $B\left(\vec{x}, \frac{1}{k}\right) \cap S \neq \varnothing$, $B\left(\vec{x}, \frac{1}{k}\right) \cap S^c \neq \varnothing$, so $\exists \vec{x}_k \in B\left(\vec{x}, \frac{1}{k}\right) \cup S.$, gives that $|\vec{x}_k - \vec{x}| < \frac{1}{k} \to 0$ as $k \to \infty$. □

Theorem 1.3 (B-W). *Every closed sequence in $\mathbb{R}$ has a convergent subsequence.*

1.5 Properties of Sets

Definition 1.10 (Compactness). *If $S \in \mathbb{R}^n$ is closed and bounded, then S is compact.*

Theorem 1.4. *$S \in \mathbb{R}^n$ is compact iff every sequence from S has a convergent subsequence with limit in S.*

Theorem 1.5. *$S \in \mathbb{R}^n$ is compact, $f : S \to \mathbb{R}^n$ is continuous, then $f(S)$ is compact.*

Proof. To see that $f(S)$ is compact, let $y_k = f(S)$ be a sequence so there are $x_k \in S$ with $f(x_k) = y_k$. There is a sequence $x_{k_l} \to x \in S$ as $l \to \infty$. By continuity, $f(x_{k_l}) \to f(x)$ as $l \to \infty$, or $y_{k_l} \to y$ as $l \to \infty$. So $f(S)$ is compact. □

Corollary 1.1. *If $f : S_{compact} \to \mathbb{R}$ is continuous, then f attains its max and min:*

$$\exists x_m, x_M \in S.\ f(x_m) \leq f(x) \leq f(x_M)$$

Proof. $f(S)$ is compact, closed and bounded, meaning $\sup f(S)$ and $\inf f(S)$ exists. Since $f(S)$ is closed, $\sup f(S) \in f(S)$ and $\inf f(S) \in f(S)$. □

Definition 1.11 (Connectness). *S is disconnected means $S = U \cup V$, where U and V are nonempty, $\bar{U} \cap V = \varnothing$ and $U \cap \bar{V} = \varnothing$. If S is not disconnected, then S is connected.*

Definition 1.12 (Interval in $\mathbb{R}$). *$I \in \mathbb{R}$ is an interval means if $a, b \in I$, $a < b$ then $(a, b) \subseteq I$.*

Theorem 1.6. *The connected sets in $\mathbb{R}$ are intervals.*

Theorem 1.7. *$S \in \mathbb{R}^n$ is connected. $f : S \to \mathbb{R}^m$ is continuous. Then $f(S)$ is connected.*

Proof. If $f(S)$ is not connected, there's disjoint set U and V in $f(S)$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint in S and $S = f^{-1}(U) \cup f^{-1}(V)$, which contradicts the connectedness of S. □

Definition 1.13 (Path Connectness). $\forall a, b \in S.\ \exists f_{cont} : [0, 1] \to S.\ f(0) = a, f(1) = b$

Theorem 1.8. *If S is connected and open in $\mathbb{R}^n$, then S is path connected.*

Theorem 1.9 (Extreme Value Theorem). *$K \subseteq \mathbb{R}^n$ compact, $f : K \to \mathbb{R}$, then f attains its maximum, or*

$$\exists \vec{X}_m, \vec{x}_m \in K.\ \forall \vec{x} \in K.\ f(\vec{x}_m) \leq f(\vec{x}) \leq f(\vec{X}_m)$$

Theorem 1.10 (Intermediate Value Theorem). *$S \subseteq \mathbb{R}^n$ is connected, $f : S \to \mathbb{R}$ is continuous, then for any $a, b \in S$ and any value c between $f(a), f(b)$, there is $x \in S$ with $f(x) = c$.*

Ex 1.4. *$S \subseteq \mathbb{R}^2$ connected, $(-1,2) \in S$, $(3,1) \in S$. Show that there is a point on the graph that is in S.*

Proof. Define $f(x,y) = y - x^3$ is continuous on S connected. $f(-1,2) = 3, f(3,1) = -26$. By IVT, $\exists (x,y) \in S.\ f(x,y) = 0, y - x^3 = 0$. □

Theorem 1.11. *$S = \{|\vec{x}| = 1\}, f : S \to \mathbb{R}$ continuous on the sphere, then*

$$\exists \vec{x} \in S.\ f(\vec{x}) = f(-\vec{x})$$

Proof. Set $g(\vec{x}) = f(\vec{x}) - f(-\vec{x})$ is continuous on S. If there is a point $\vec{p} \in S$ with $g(\vec{p}) > 0$,

$$g(-\vec{p}) = f(-\vec{p}) - f(\vec{p}) = -(f(\vec{p}) - f(-\vec{p})) = -g(\vec{p}) < 0$$

By the IVT, there exists $\vec{x} \in S$ with $g(\vec{x}) = 0$, therefore $f(\vec{x}) = f(-\vec{x})$. □

Definition 1.14 (Uniform Continuity). *$S \subseteq \mathbb{R}^m$, $f : S \to \mathbb{R}^n$, f is uniformly continuous on S means*

$$\forall \epsilon > 0.\ \exists \delta > 0.\ \forall \vec{x} \in S.\ f(B(\vec{x}, \delta) \cap S) \subseteq B(f(\vec{x}), \epsilon)$$

Ex 1.5. *Prove if $|f(x) - f(y)| \leq 10\,|x-y|^{\frac{1}{2}}$ for all $x, y \in S$ then f is unif. cont on S.*

Proof. Note

$$\begin{aligned} |x - y| < \delta \Rightarrow 10\,|x-y|^{\frac{1}{2}} &< 10\delta^{\frac{1}{2}} \\ 10\delta^{\frac{1}{2}} &< \epsilon \\ \delta &< \left(\frac{\epsilon}{10}\right)^2 \end{aligned}$$

Given $\epsilon > 0$ let $0 < \delta < \left(\frac{\epsilon}{10}\right)^2$. For $x, y \in S$ with $|x - y| < \delta$ we get

$$\begin{aligned} |f(x) - f(y)| \leq 10\,|x-y|^{\frac{1}{2}} &< 10\delta^{\frac{1}{2}} \\ 10 \cdot \frac{\epsilon}{10} &= \epsilon \end{aligned}$$

□

Remark 1.2 (Lipschitz Functions). *If $f : S \to \mathbb{R}^m$ satisfies $\forall x, y \in S.\ |f(x) - f(y)| \leq C\,|x - y|$, then f is uniformly continuous.*

Theorem 1.12. *If $S \subseteq \mathbb{R}^m$ is compact and $f : S \to \mathbb{R}^m$ is continuous, then f is uniformly continuous.*

Lemma 1.1 (Heine Borel). *Every cover of S by open sets has a finite subcover.*

Proof. Given $\epsilon > 0$, by continuity at x, for every $x \in S$ there is a ball $B(x, \delta_x)$ so that

$$\forall y \in B(x, \delta_x).\ |f(x) - f(y)| < \epsilon$$

Now $S \subseteq \bigcup_{x \in S} B\left(x, \frac{\delta x}{2}\right)$, by H.B,

$$\exists x_1, \ldots, x_l.\ S \subseteq \bigcup_{i=1}^{l} B\left(x_i, \frac{\delta_i}{2}\right)$$

Let $x, y \in S$ with $|x - y| < \delta = \min_{i=1,\ldots,l} {}^{\delta_{x_i}}/2$. Now $x \in S$ so $x \in B(x_i, {}^{\delta_{x_i}}/2)$

$$\begin{aligned} |y, x_i| &\leq |y - x| + |x - x_i| \\ &< \frac{\delta_{x_i}}{2} + \frac{\delta_{x_i}}{2} = \delta_{x_i} \\ y \in B(x_i, \delta_{x_i})\, |f(x) - f(y)| &< |f(x) - f(x_i)| + |f(y) - f(x_i)| \\ &< \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

□

Chapter 2

Differential Calculus

2.1 One variable

Definition 2.1 (Derivative). *I be an open interval in $\mathbb{R}$, $a \in I$, $f : I \to \mathbb{R}$, f is differentiable at a means*

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{n} \quad \textit{exists}$$

this is equivalent to

$$\exists m.\ f(a+h) = f(a) + mh + o(h)$$

where $m = f'(a)$ and $o(h)$ means $\lim_{h \to 0} \frac{o(h)}{h} = 0$. Think of h as the variable, then $f(a+h)$ is approx $f(a)+mh$

Theorem 2.1 (Product Rule). *If f, g are differentiable at a, then $f \cdot g$ is differentiable at a and*

$$(f \cdot g)'(a) = (f' \cdot g)(a) + (f \cdot g')(a)$$

Proof. We know

$$\begin{aligned}
f(a+h) &= f(a) + f'(a)h + o(h) \\
g(a+h) &= g(a) + g'(a)h + o(h) \\
(f \cdot g)(a+h) &= f(a+h)g(a+h) \\
&= (f(a) + f'(a)h + o(h))(g(a) + g'(a)h + o(h)) \\
&= f(a)g(a) + f(a)g'(a)h + f(a)o(h) \\
&\quad + f'(a)g(a)h + f'(a)g'(a)h^2 + f'(a)o(h)h \\
&\quad + o(h)g(a) + o(h)g'(a)h + o(h)o(h) \\
&= f(a)g(a) + (f(a)g'(a) + f'(a)g(a))h + o(h)
\end{aligned}$$

□

Theorem 2.2. *I open interval, $a \in I$, $f : I \to \mathbb{R}$.*

1. *f has a local maximum at a means $\forall x \in B(a, \delta).\ \ f(x) \le f(a)$, and $f'(a) = 0$.*

   Proof.

$$\begin{aligned}
f(a+h) &= f(a) + f'(a)h + o(h) \\
f(a+h) - f(a) &= f'(a)h + o(h) \le 0 && \text{(for } |h| < \delta) \\
f'(a) + \frac{o(h)}{h} &\le 0 && \text{(h > 0)} \\
f'(a) + \frac{o(h)}{h} &\ge 0 && \text{(h < 0)}
\end{aligned}$$

With $h \to 0$

$$f'(a) \leq 0, \quad f'(a) \geq 0$$
$$f'(a) = 0$$

□

Theorem 2.3 (Rolle's)**.** *$f : [a, b] \to \mathbb{R}$ continuous, f differentiable on (a, b) and $f(a) = f(b)$ then $\exists c \in [a, b].\ f'(c) = 0$*

Proof. By the EVT, f has a max and min at x_M, x_m. If $x_M > f(a)$, then f has a local maximum at x_M, so $c = f'(x_M) = 0$. If $x_m < f(a)$, then f has a local minimum at x_m, so $c = f'(x_m) = 0$. Otherwise, f is constant, $\forall x \in (a, b).\ f'(x) = 0$. □

Theorem 2.4 (Mean Value Theorem)**.**

Proof. $f : [a, b] \to \mathbb{R}$ continuous and differentiable on (a, b), then

$$\exists c \in (a, b).\ f'(c) = \frac{f(b) - f(a)}{b - a}$$

or

$$\exists c \in (a, b).\ f'(c)(b - a) = f(b) - f(a)$$

□

Corollary 2.1. *f is differentiable on (a, b)*

1. *If $\forall x \in (a, b).\ f'(x) \geq 0$ then f is increasing on (a, b).*
2. *If $\forall x \in (a, b).\ f'(x) \leq 0$ then f is decreasing on (a, b).*
3. *If $\forall x \in (a, b).\ f'(x) > 0$ then f is strictly increasing on (a, b).*
4. *If $\forall x \in (a, b).\ f'(x) < 0$ then f is strictly decreasing on (a, b).*
5. *If $\forall x \in (a, b).\ f'(x) = 0$ then f is constant on (a, b).*

Theorem 2.5. *If f' is bounded on interval $S \subseteq \mathbb{R}$ then f uniformly continuous on S.*

Proof. For $x, y \in S$, apply the MVT to f on $[x, y]$

$$\exists M.\ \forall x \in S.\ |f'(x)| \leq M$$
$$\exists x \in (x, y).\ f(y) - f(x) = f'(c)(y - x)$$
$$|f(y) - f(x)| = |f'(c)|\,|y - x|$$
$$\leq M\,|x - y|$$

f is Lipschitz, so f is uniformly continuous. □

Theorem 2.6 (Generalized Mean Value Theorem)**.** *$f, g : [a, b] \to \mathbb{R}$ continuous and differentiable on (a, b), then*

$$\exists c \in (a, b).\ (f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$$

Theorem 2.7 (L'Hospital's Rule)**.** *$f, g : [a, b] \to \mathbb{R}$ continuous and differentiable on (a, b), $g'(x) \neq 0$ on (a, b), then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Ex 2.1. $f(x) = \begin{cases} x^{2+a}\sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$*, find $f'(x)$ for all $x \in \mathbb{R}$.*

Answer. For $x \neq 0$,

$$\begin{aligned} f'(x) &= (2+a)x^{1+a}\sin\left(\frac{1}{x}\right) - x^{2+a}\cos\left(\frac{1}{x}\right)\left(\frac{1}{x^2}\right) \\ &= (2+a)x^{1+a}\sin\left(\frac{1}{x}\right) - x^{a}\cos\left(\frac{1}{x}\right) \end{aligned}$$

for $x = 0$

$$\begin{aligned} f'(0) &= \lim_{h\to 0}\frac{f(h)-f(0)}{h} \\ &= \lim_{h\to 0}\frac{h^{2+a}\sin\left(\frac{1}{h}\right)}{h} \\ &= \lim_{h\to 0} h^{1+a}\sin\left(\frac{1}{h}\right) = 0 && \text{(By Squeeze Theorem)} \end{aligned}$$

Check for continuity of f' at 0

$$\begin{aligned} \lim_{x\to 0} f'(x) &= \lim_{x\to 0}(2+a)x^{1+a}\sin\left(\frac{1}{x}\right) - x^{a}\cos\left(\frac{1}{x}\right) \\ &= 0 \end{aligned}$$

Theorem 2.8. *If f is differentiable on a set, then it's continuous on that set, and $f \in C^1$.*

Theorem 2.9. *f is twice differentiable on I, $a \in I$. Prove that*

$$\lim_{h\to 0}\frac{f(a+2h)-2f(a+h)+f(a)}{h^2} = f''(a)$$

Proof.

$$\begin{aligned} \lim_{h\to 0}\frac{f(a+2h)-2f(a+h)+f(a)}{h^2} &= \lim_{h\to 0}\frac{2f'(a+2h)-2f'(a+h)}{2h} \\ &= \lim_{h\to 0}\frac{f'(a+2h)-f'(a+h)}{h} \\ &= \lim_{h\to 0} 2f''(a+2h)-f''(a+h) \\ \lim_{h\to 0}\frac{f'(a+2h)-f'(a+h)}{h} &= \lim_{h\to 0}\frac{f'(a+2h)+f(a)}{h} + \frac{f(a)-f'(a+h)}{h} \\ &= \lim_{h\to 0} 2\cdot\frac{f'(a+2h)+f(a)}{2h} - \frac{f'(a+h)-f(a)}{h} \\ &= 2f''(a)-f''(a) \\ &= f''(a) \end{aligned}$$

□

Ex 2.2. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

Answer.

$$\begin{aligned}
\log\left(1+\frac{1}{x}\right)^x &= x\log\left(1+\frac{1}{x}\right)\\
&= \frac{\log\left(1+\frac{1}{x}\right)}{\frac{1}{x}}\\
\lim_{x\to\infty}\log\left(1+\frac{1}{x}\right)^x &= \lim_{x\to\infty}\frac{1}{\left(1+\frac{1}{x}\right)}\\
&= \lim_{x\to\infty}\frac{x}{x+1}\\
&= 1
\end{aligned}$$

2.2 Vector Value Functions

Definition 2.2.

$$\begin{aligned}
\vec{f} &: I \to \mathbb{R}^n\\
\vec{f}(t) &= (f_1(t),\ldots,f_n(t))\\
\vec{f}'(t) &= (f_1'(t),\ldots,f_n'(t))
\end{aligned}$$

Theorem 2.10 (Product Rule)**.**

$$\begin{aligned}
((\vec{f}\cdot\vec{g})(t))' &= \left(\sum_{i=1}^n f_i(t)g_i(t)\right)'\\
&= \sum_{i=1}^n (f_i(t)g_i(t))'\\
&= \sum_{i=1}^n f_i'(t)g_i(t) + f_i(t)g_i'(t)\\
&= \sum_{i=1}^n f_i'(t)g_i(t) + \sum_{i=1}^n f_i(t)g_i'(t)\\
&= (f'g)(t) + (fg')(t)
\end{aligned}$$

Definition 2.3. $\vec{f},\vec{g}: I\to\mathbb{R}^3$, $\vec{f}=(f_1,f_2,f_3), \vec{g}=(g_1,g_2,g_3)$,

$$\begin{aligned}
\vec{f}\times\vec{g} &= \vec{e}_1(f_2g_3-f_3g_2)+\vec{e}_2(f_3g_1-f_1g_3)+\vec{e}_3(f_1g_2-f_2g_1)\\
&= (f_2g_3-f_3g_2, f_3g_1-f_1g_3, f_1g_2-f_2g_1)\\
(\vec{f}\times\vec{g})' &= ((f_2g_3-f_3g_2)', (f_3g_1-f_1g_3)', (f_1g_2-f_2g_1)')\\
(f_2g_3-f_3g_2)' &= f_2'g_3+f_2g_3'-f_3'g_2-f_3g_2'\\
&\vdots\\
(\vec{f}\times\vec{g})' &= \vec{f}'\times\vec{g}+\vec{f}\times\vec{g}'
\end{aligned}$$

2.3 Partial Derivative

Definition 2.4 (Partial Derivative). $f : \mathbb{R}^n \to \mathbb{R}$, $f(x_1, x_2, \ldots, x_n)$.

$$\frac{\partial}{\partial x_j} f(\vec{x}) = \lim_{h\to 0} \frac{f(\vec{x} + h\vec{e}_j) - f(\vec{x})}{h}$$

Ex 2.3. *Find* $\partial_x \frac{e^{xyz}}{x^2+y^2+z^2}$.

Answer.

$$\frac{(x^2 + y^2 + z^2)e^{xyz}yz - e^{xyz}2x}{(x^2 + y^2 + z^2)^2}$$

Ex 2.4. $f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$

Answer.

$$\partial_x f(0, 0) = \lim_{h\to 0} \frac{f(0 + h) - f(0, 0)}{h} = 0$$
$$\partial_y f(0, 0) = \lim_{h\to 0} \frac{f(0 + h) - f(0, 0)}{h} = 0$$

But $f(x, y)$ is not continuous at $(0, 0)$.

Definition 2.5 (Gradient). $S \subseteq \mathbb{R}^n$*, S is open,* $\vec{a} \in S$*.* $f : S \to \mathbb{R}$*, f is differentiable at* $\vec{a}$ *means*

$$\exists \vec{c} \in \mathbb{R}^n.\ f(\vec{a} + \vec{h}) = \underbrace{f(\vec{a}) + \vec{c} \cdot \vec{h}}_{\text{linear in } h_1,\ldots,h_n} + o(\vec{h})$$

Then $\vec{c} = \nabla f(\vec{a})$.

Theorem 2.11. *If* $\vec{f}$ *is differentiable at* $\vec{a}$ *then*

$$\nabla f(\vec{a}) = (\partial_{x_1} f(a), \ldots, \partial_{x_n} f(a))$$

Proof.

$$\begin{aligned}
\partial_{x_j} f(a) &= \lim_{h\to 0} \frac{f(\vec{a} + h\vec{e}_j) - f(\vec{a})}{h} \\
&= \lim_{h\to 0} \frac{\nabla f(\vec{a}) \cdot h\vec{e}_j + o(h\vec{e}_j)}{h} \\
&= \nabla f(\vec{a}) \cdot \vec{e}_j + \lim_{h\to 0} \frac{o(h\vec{e}_j)}{h} \\
&= \nabla f(\vec{a}) \cdot \vec{e}_j
\end{aligned}$$

□

Theorem 2.12 (Chain Rule). *Let* $\vec{g}(t) : \mathbb{R} \to \mathbb{R}^n, f(x) : \mathbb{R}^n \to \mathbb{R}, f \circ \vec{g}(t) = f(\vec{g}(t)) : \mathbb{R} \to \mathbb{R}$*. If* $\vec{g}$ *is differentiable at* a *and* f *is differentiable at* $b = \vec{g}(a)$*, then* $f \circ \vec{g}$ *is differentiable at* a *and*

$$(f \circ \vec{g})'(a) = \nabla f(b) \cdot \vec{g}'(a)$$

Proof.

$$\begin{aligned}
\vec{g}(a + h) &= \vec{g}(a)\vec{g}'(a)h + o(h) \\
f(\vec{b} + \vec{k}) &= f(\vec{b}) + \nabla f(\vec{b}) \cdot \vec{k} + o(\vec{k}) \\
(f \circ \vec{g})(a + h) &= f(\vec{g}(a + h)) \\
&= f(\underbrace{\vec{g}(a)}_{\vec{b}} + \underbrace{\vec{g}'(a)h + o(h)}_{\vec{k}}) \\
&= \underbrace{f(\vec{b})}_{(f\circ\vec{g})(a)} + \underbrace{\nabla f(\vec{b}) \cdot \vec{g}'(a)h}_{(f\circ\vec{g})'(a)} + \nabla f(\vec{b}) \cdot o(h) + o(h)
\end{aligned}$$

Recall that $o(\vec{k}) = e(\vec{k})\left|\vec{k}\right|$ where $e(\vec{k}) \to 0$ as $\vec{k} \to 0$

$$\lim_{h\to 0} \frac{e(\vec{k})\left|\vec{k}\right|}{h} = 0$$

$\square$

Ex 2.5. *$S = \{\vec{x} \in \mathbb{R}^n \mid F(\vec{x}) = 0\}$. It's a surface in R^n. Take $\vec{a} \in S, F(\vec{a}) = 0$. Take any curve in S through $\vec{a}$. Take $\vec{g}(t), \vec{g}(t) = \vec{a}, t \in [-1, 1]$, then for all $t \in [-1, 1]$, $F(\vec{g}(t)) = 0$.*

$$\begin{aligned}\frac{d}{dt}F(\vec{g}(t)) &= 0\\ \nabla F(\vec{g}(t)) \cdot \vec{g}'(t) &= 0\\ \nabla F(\vec{a}) \cdot \vec{g}'(t) &= 0\end{aligned}$$

$\vec{g}'(t)$ is orthogonal to $\nabla F(\vec{a})$, where $g'(t)$ forms a tangent plane, and $\nabla F(\vec{a})$ is normal to the tangent plane, or S at $\vec{a}$.

Ex 2.6. $x^2 + \frac{y^2}{9} + \frac{z^2}{16} = 1$. *What is the normal to the ellipse at the point $(0, 0, 4)$?*

Answer.

$$\begin{aligned}\underbrace{x^2 + \frac{y^2}{9} + \frac{z^2}{16} - 1}_{F(x,y,z)} &= 0\\ \nabla F(0,0,4) &= \nabla F(0,0,4)|_{(0,0,4)}\\ &= \left(2x, \frac{2y}{9}, \frac{z}{8}\right)\Big|_{(0,0,4)}\\ &= \left(0, 0, \frac{1}{2}\right)\end{aligned}$$

And the tangent plane will be

$$\begin{aligned}\nabla F(\vec{a}) \cdot (\vec{x} - \vec{a}) &= 0\\ \left(0, 0, \frac{1}{2}\right) \cdot (x - 0, y - 0, z - 4) &= 0\\ \frac{z-4}{2} &= 0\\ z &= 4\end{aligned}$$

Recall 2.1. *Line segment from $\vec{a}$ to $\vec{b}$.*

$$\begin{aligned}\vec{l}(t) &= (1-t)\vec{a} + t\vec{b} \quad 0 \le t \le 1\\ &= \vec{a} + t(\vec{b} - \vec{a})\\ \vec{l}'(t) &= \vec{b} - \vec{a}\end{aligned}$$

Definition 2.6 (Convex set)**.** *$S \subseteq \mathbb{R}^n$, S is convex means*

$$\forall \vec{a}, \vec{b} \in S.\ L_{\vec{a}\vec{b}} \subseteq S$$

Ex 2.7. *Balls are convex*

Theorem 2.13. *If S is convex, open and f is differentiable on S with*

$$\forall \vec{x} \in S.\ |\nabla f(\vec{x})| \leq M$$

then

$$\forall \vec{x}, \vec{u}.\ |f(\vec{x}) - f(\vec{y})| \leq M\,|\vec{x} - \vec{y}|$$

Proof. Let $\vec{x}, \vec{y} \in S, \vec{l}(t) = \vec{x} + t(\vec{y} - \vec{x}), 0 \leq t \leq 1$ $f(\vec{l}(t))$ is diff on $(0,1)$, cont on $[0,1]$. By the MVT,

$$\begin{aligned} f(\vec{l}(1)) - f(\vec{l}(0)) &= \frac{d}{dt} f(\vec{l}(t))|_{t=t_c}(1-0) \\ \frac{d}{dt} f(\vec{l}(t)) &= \nabla f(\vec{l}(t)) \cdot \vec{l}'(t) \\ &= \nabla f(\vec{l}(t)) \cdot (\vec{y} - \vec{x}) \\ |f(\vec{y}) - f(\vec{x})| &= \nabla f(\vec{l}(t)) \cdot (\vec{y} - \vec{x}) \\ &\leq \left| \nabla f(\vec{l}(t)) \cdot (\vec{y} - \vec{x}) \right| \\ &\leq M\,|(\vec{y} - \vec{x})| \end{aligned}$$

□

Theorem 2.14. *If S is open, connected and $\nabla f(\vec{x}) = 0$ for all $\vec{x} \in S$ then $f(\vec{x})$ is constant.*

Proof. For any line segment from $\vec{a}$ to $\vec{b}$ in S by previous theorem.

$$\begin{aligned} \left| f(\vec{b}) - f(\vec{a}) \right| &\leq 0 \cdot \left| \vec{b} - \vec{a} \right| \\ \implies \quad f(\vec{a}) &= f(\vec{b}) \end{aligned}$$

Open connected $S \in \mathbb{R}^n$ are path connected, even step path connected. □

Theorem 2.15 (Implicit function theorem)**.** *One Equation*

$$F(x_1, x_2, \ldots, x_n, y) = 0$$

think $y(\vec{x})$ as a function of x, then

$$\begin{aligned} \frac{\partial F}{\partial x_1} = \sum_{i=1}^{n} (\partial_{x_i} F) \left(\frac{\partial x_i}{x_1} \right) + (\partial_y F) \left(\frac{\partial y}{x_1} \right) &= 0 \\ \partial_{x_1} F + \frac{\partial_y F \partial y}{x_1} &= 0 \\ F_{x_1} + y_{x_1} F_y &= 0 \\ y_{x_1} &= \frac{-F_{x_1}}{F_y} \end{aligned}$$

Ex 2.8.

$$\begin{aligned} x^2 + y^2 + z^2 &= 1 \\ \underbrace{x^2 + y^2 + z^2 - 1}_{F(x,y,z)} &= 0 \end{aligned}$$

think $x(y, z)$

$$\begin{aligned} \frac{\partial x}{\partial y} = \frac{-F_y}{F_x} &= -\frac{2y}{2x} = -\frac{y}{x} \\ \frac{\partial x}{\partial x} &= -\frac{z}{x} \end{aligned}$$

think $y(x,z)$

$$\frac{\partial y}{\partial x} = \frac{-F_x}{F_y} = -\frac{2x}{2y} = -\frac{x}{y}$$

Theorem 2.16 (Two equations in 4 unknowns)**.**

$$\begin{aligned} F(x,y,u,v) &= 0 \\ G(x,y,u,v) &= 0 \end{aligned}$$

Think $u(x,y), v(x,y)$*, find* u_x, u_y, v_x, v_y*. Take* ∂_x *of both equations.*

$$\begin{aligned} F_x + F_u u_x + F_v v_x &= 0 \\ G_x + G_u u_x + G_v v_x &= 0 \\ \begin{bmatrix} F_x \\ G_x \end{bmatrix} + \begin{bmatrix} F_u & F_v \\ G_u & G_v \end{bmatrix} \begin{bmatrix} u_x \\ v_x \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} F_u & F_v \\ G_u & G_v \end{bmatrix} \begin{bmatrix} u_x \\ v_x \end{bmatrix} &= -\begin{bmatrix} F_x \\ G_x \end{bmatrix} \\ u_x &= -\frac{\begin{vmatrix} F_x & F_u \\ G_x & G_u \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} \\ v_x &= -\frac{\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} \\ \frac{\partial u}{\partial x} = u_x &= \frac{-\frac{\partial(F,G)}{\partial(x,v)}}{\frac{\partial(F,G)}{\partial(u,v)}} \\ \frac{\partial v}{\partial x} = v_x &= \frac{-\frac{\partial(F,G)}{\partial(u,x)}}{\frac{\partial(F,G)}{\partial(u,v)}} \\ \frac{\partial u}{\partial y} = u_y &= \frac{-\frac{\partial(F,G)}{\partial(y,v)}}{\frac{\partial(F,G)}{\partial(u,v)}} \\ \frac{\partial v}{\partial y} = v_y &= \frac{-\frac{\partial(F,G)}{\partial(u,y)}}{\frac{\partial(F,G)}{\partial(u,v)}} \end{aligned}$$

Ex 2.9.

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ F(x,y,r,\theta) = x - r\cos\theta &= 0 \\ G(x,y,r,\theta) = y - r\sin\theta &= 0 \end{aligned}$$

What are $\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}$*?*

$$\begin{aligned}
\frac{\partial r}{\partial x} = \frac{-\frac{\partial(F,G)}{\partial(x,\theta)}}{\frac{\partial(F,G)}{\partial(r,\theta)}} &= -\frac{\begin{vmatrix} 1 & r\sin\theta \\ 0 & -r\cos\theta \end{vmatrix}}{\begin{vmatrix} -\cos\theta & r\sin\theta \\ -\sin\theta & -r\cos\theta \end{vmatrix}} \\
&= \frac{r\cos\theta}{r\cos^2\theta + r\sin^2\theta} \\
&= \frac{r\cos\theta}{r} \\
&= \cos\theta \\
&= \frac{x}{\sqrt{x^2+y^2}} \\
\frac{\partial r}{\partial y} &= \frac{\begin{vmatrix} 0 & r\sin\theta \\ 1 & -r\cos\theta \end{vmatrix}}{r} \\
&= \frac{r\sin\theta}{r} \\
&= \sin\theta \\
\frac{\partial \theta}{\partial x} = \frac{-\frac{\partial(F,G)}{\partial(r,x)}}{\frac{\partial(F,G)}{\partial(r,\theta)}} &= \frac{-\begin{vmatrix} -\cos\theta & 1 \\ -\sin\theta & 0 \end{vmatrix}}{r} \\
&= \frac{-\sin\theta}{r} \\
&= \frac{-r\sin\theta}{r^2} \\
&= -\frac{y}{x^2+y^2} \\
\frac{\partial \theta}{\partial y} = \frac{-\frac{\partial(F,G)}{\partial(r,y)}}{\frac{\partial(F,G)}{\partial(r,\theta)}} &= \frac{-\begin{vmatrix} -\cos\theta & 0 \\ -\sin\theta & 1 \end{vmatrix}}{r} \\
&= \frac{-\cos\theta}{r} \\
&= \frac{r\cos\theta}{r^2} \\
&= \frac{x}{x^2+y^2} \\
\nabla\theta(x,y) &= \langle \theta_x, \theta_y \rangle \\
&= \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle \\
&= \frac{1}{r^2}\langle -y, x \rangle \\
\nabla\theta(x,y)\cdot(x,y) &= 0
\end{aligned}$$

2.4 Higher Order Partial Derivative

Ex 2.10.

$$u = xe^{yz}$$
$$\partial_x u = e^{yz}$$
$$u_{xy} = \partial_y \partial_x u = ze^{yz} = \partial_x \partial_y u$$

Ex 2.11. *There is an $f(x, y)$ so that mixed partials can* ***depend on*** *the order, i.e.,*

$$\partial_y \partial_x f(0,0) \neq \partial_x \partial_y f(0,0)$$

Theorem 2.17. *f on $S \subseteq \mathbb{R}$, open, $a \in S$. All second order partial derivatives of f, $\partial_i f, \partial_i \partial_j f, \partial_j \partial_i f$ exists in S and if $\partial_i \partial_j f$, $\partial_j \partial_i f$ are continuity at $\vec{a}$, then*

$$\partial_i \partial_j f(\vec{a}) = \partial_j \partial_i f(\vec{a})$$

if $f \in C^k(S)$, then any k derivative are not depend on the order.

Remark 2.1. *$f \in C^2(S)$ then any mixed partial do not depend on the order of differentiation.*

Definition 2.7 (Laplacian in $\mathbb{R}^n$)**.** *$u : S \to \mathbb{R}$, S open, $u \in C^2(S)$, then*

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i}$$

Solutions to $\Delta u = 0$, u is called harmonic.

Ex 2.12. *In $\mathbb{R}^2$, $u_{xx} + u_{yy} = 0$; in $\mathbb{R}^3$, $u_{xx} + u_{yy} + u_{zz} = 0$. Rewrite in polar coordinate and spherical coordinate.*
Idea: *$f(x, y), f_{xx} + f_{yy}$, but $x = r\cos\theta, y = r\sin\theta, u = f(r\cos\theta, r\sin\theta)$.*

$$\begin{aligned}
u_r &= f_x \cdot \frac{\partial x}{\partial r} + f_y \cdot \frac{\partial y}{\partial r} \\
&= f_x \cos\theta + f_y \sin\theta \\
u_{rr} &= (f_{xx}\cos\theta + f_{xy}\sin\theta)\cos\theta + (f_{yx}\cos\theta + f_{yy}\sin\theta)\sin\theta \\
u_\theta &= f_x \cdot \frac{\partial x}{\partial \theta} + f_y \cdot \frac{\partial y}{\partial \theta} \\
&= -r\cos\theta f_x + r\cos\theta f_y \\
u_{r\theta} &= -\sin\theta f_x + \cos\theta f_y + \cos\theta (f_x)_\theta + \sin\theta (f_y)_\theta \\
&= -\sin\theta f_x + \cos\theta f_y + \cos\theta(-r\sin\theta f_{xx} + r\cos\theta f_{xy}) + \sin\theta(-r\sin\theta f_{xy} + r\cos\theta f_{yy}) \\
u_{\theta\theta} &= -r\cos\theta f_x - r\sin\theta f_y - r\sin\theta (f_x)_\theta + r\cos\theta (f_y)_\theta \\
&= -r\cos\theta f_x - r\sin\theta f_y - r\sin\theta(-r\sin\theta f_{xx} + r\cos\theta f_{xy}) + r\cos\theta(-r\sin\theta f_{yx} + r\cos\theta f_{yy})
\end{aligned}$$

$$f_{xx} + f_{yy} = u_{rr} + \frac{u_{\theta\theta}}{r^2} + \frac{1}{r} u_r \qquad \text{(Polar Coordinate)}$$

Recall. *In polar, $x = r\cos\theta, y = r\sin\theta$. In spherical, $x = r\sin\phi\cos\theta, y = r\sin\phi\sin\theta, z = r\cos\phi$.*

$$\begin{aligned}
u &= f(x,y,z) \quad x = r\cos\theta, y = r\sin\theta, z = z \\
f_{xx} + f_{yy} + f_{zz} &= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} && \text{(Cylindrical Coordinate)} \\
z &= \rho\cos\phi \quad r = \rho\sin\phi \\
\tan\phi &= \frac{r}{z} \\
\phi &= \tan^{-1}\left(\frac{r}{z}\right) \\
\rho &= \sqrt{r^2+z^2} \quad \frac{\partial\rho}{\partial r} = \frac{r}{\rho} \\
u_r &= w_\rho\frac{\partial\rho}{\partial\rho} + w_\theta\frac{\partial\theta}{\partial r} \\
&= w_\rho\frac{r}{\rho} + w_\phi\frac{z}{z^2+r^2} \\
&= w_\rho\sin\phi + w_\phi\frac{\cos\phi}{\rho} \\
\frac{u_r}{r} &= \frac{w_\rho\sin\phi + w_\phi\frac{\cos\phi}{\rho}}{\rho\sin\phi} \\
&= \frac{1}{\rho}w_\rho + \frac{1}{\rho^2}w_\phi\cot\phi \\
f_{xx} + f_{yy} + f_{zz} &= w_{\rho\rho} + \frac{1}{\rho^2}w_{\phi\phi} + \frac{2}{\rho}w_\rho + \frac{1}{\rho^2\sin^2\phi}w_{\theta\theta} + \frac{1}{\rho^2}w_\phi\cot\phi && \text{(Spherical Coordinate)}
\end{aligned}$$

Definition 2.8 (Multi-Index Notation). $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)$, *each* α_i *is an index*

$$\partial^{\vec{\alpha}} = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\ldots\partial_{x_n}^{\alpha_n}$$

$$\vec{\alpha}! = \alpha_1!\alpha_2!\ldots\alpha_n!$$

is an $|\vec{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ *derivative. For* $\vec{x} \in \mathbb{R}^n$*,* $\vec{x} = (x_1, x_2, \ldots, x_n)$

$$\vec{x}^{\vec{\alpha}} = x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n}$$

Ex 2.13. $\vec{a} = (3,2,1)$*,* $f(x,y,z) = x^3y^4z$.

$$\begin{aligned}
\partial^{\vec{a}} &= \partial_x^3\partial_y^2\partial_z x^3y^4z \\
&= \partial_x^3\partial_y^2 x^3y^4 \\
&= \partial_x^3 12x^3y^2 \\
&= 72y^2
\end{aligned}$$

Ex 2.14. *Binomial expansion:*

$$(x_1+x_2)^k = \sum_{j=0}^{j}\binom{k}{j}x_1^jx_2^{k-j}$$

Let $\vec{\alpha} = (j, k-j)$*, then*

$$\binom{k}{j} = \frac{k!}{(k-j)!j!} = \frac{k!}{\vec{\alpha}!}$$

$$(x_1+x_2)^k = \sum_{|\vec{\alpha}|=k}\frac{k!}{\vec{a}!}\vec{x}^{\vec{\alpha}}$$

Which is also true for $\vec{x} \in \mathbb{R}^n$

$$(x_1 + x_2 + \cdots + x_n)^k = \sum_{|\vec{\alpha}|=k} \frac{k!}{\vec{a}!} \vec{x}^{\vec{\alpha}}$$

Ex 2.15. $f(x, y, z)$

$$\begin{aligned}
|\vec{\alpha}| &= 2 \\
\vec{\alpha} &= (2,0,0), (0,2,0), (0,0,2), (1,1,0), (1,0,1), (0,1,1) \\
\sum_{|\vec{a}|=2} \frac{\partial^{\vec{a}} f(0)}{\vec{a}!} \vec{x}^{\vec{a}} &= \frac{\partial_x^2 f(0)}{2!} x^2 + \frac{\partial_y^2 f(0)}{2!} y^2 + \frac{\partial_z^2 f(0)}{2!} Z + \partial_x \partial_y f(0) xy + \partial_x \partial_z f(0) xz + \partial_y \partial_z f(0) yz \\
&= \frac{1}{2!} f_{xx} x^2 + \frac{1}{2!} f_{yy} y^2 + \frac{1}{2!} f_{zz} z^2 + f_{xy} xy + f_{xz} xz + f_{yz} yz \\
&= \frac{1}{2!} [f_{xx} x^2 + f_{yy} y^2 + f_{zz} z^2 + 2 f_{xy} xy + 2 f_{xz} xz + 2 f_{yz} yz] \\
&= \frac{1}{2!} \vec{x}^T D^2 f \vec{x} \\
&= \frac{1}{2!} [x \; y \; z] \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}
\end{aligned}$$

2.5 Taylor's Theorem

Theorem 2.18 (Taylor's). *$f : I -> \mathbb{R}, a \in I, f \in C^k(I)$, then*

$$\begin{aligned}
f(a+h) &= P_{a,k}(h) = R_{a,k}(h) \\
P_{a,k}(h) &= \sum_{j=0}^{k} \frac{f^{(j)}(a)}{j!} h^j \\
R_{a,k}(h) &= o\left(h^k\right)
\end{aligned}$$

If $f \in C^{k+1}(I)$ then there exists c between o and h such that

$$R_{a,k}(h) = \frac{f^{(k+1)}(a+c)}{(k+1)!} h^{k+1}$$

Remark 2.2. *If $f \in C^1$*

$$f(a+h) = \underbrace{f(a) + f'(a)h}_{P_{a,1}(h)} + \underbrace{o(h)}_{R_{a,1}(h)}$$

If $f \in C^2$

$$f(a+h) = \underbrace{f(a) + f'(a)h + \frac{f''(a)}{2!} h^2}_{P_{a,2}(h)} + \underbrace{o(h^2)}_{R_{a,2}(h)}$$

Theorem 2.19. *If $f'(a) = 0$ and $f''(a) > 0$,*

$$f(a+h) = f(a) + \frac{f''(a)}{2!} h^2 + o(h^2)$$

$f(a+h) \geq f(a)$ for all $|h|$ small. f has a local minimum at a.

Ex 2.16 (Important Functions at $a = 0$)**.**

$$\begin{aligned}
e^x &= \underbrace{\sum_{j=0}^{10} \frac{x^j}{j!}}_{P_{0,10}(x)} + R_{0,10}(x) \\
&= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{10}}{10!} + R_{0,10}(x) \\
\sin(0+x) &= \underbrace{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}}_{P_{0,9}(x)} + R_{0,9}(x) \\
\cos(0+x) &= \underbrace{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}}_{P_{0,6}(x)} + R_{0,6}(x) \\
\frac{1}{1-x} &= \underbrace{1 + x + x^2 + \cdots + x^n}_{P_{0,n}(x)} + R_{0,n}(x)
\end{aligned}$$

Ex 2.17.

$$\begin{aligned}
&\lim_{x\to 0} \frac{x^2 - \sin(x^2)}{x^4(1-\cos x)} \\
&= \lim_{x\to 0} \frac{x^2 - \left(x^2 - \frac{(x^2)^3}{3!} + o(x^6)\right)}{x^4\left(x - \left(x - \frac{x^2}{2!} + o(x^2)\right)\right)} \\
&= \lim_{x\to 0} \frac{x^6/3! + o(x^6)}{x^6/2! + x^4 o(x^2)} \\
&= \lim_{x\to 0} \frac{1/3! + o(x^6)/x^6}{1/2! + o(x^6)/x^6} \\
&= \frac{2!}{3!} = \frac{1}{3}
\end{aligned}$$

Definition 2.9 (Lagrange Multipliers)**.** *Find maximum or minimum of some function subject to a constraint $G(\vec{x}) = 0$, which is a plane. If ∇f is not in the direction of ∇G, then part of ∇f is in the tangent plane and f increase in that direction and decrease on the opposite direction, so* ***no*** *max or min at such a point. Therefore, look for points where $\nabla f = \lambda \nabla G$.*

Ex 2.18 (Isoperimetric Problem)**.** *Find the maximum volume of a box with surface area A.*

Answer. $f(x,y,z) = V = xyz, A = 2xy + 2xz + 2yz, x, y, z > 0$.

$$\begin{aligned}
G(x,y,y) &= 2xy + 2xz + 2yz - A \\
\nabla f &= \lambda \nabla g \\
f_x = \lambda G_x &\Rightarrow yz = 2\lambda(y+z) \\
f_y = \lambda G_y &\Rightarrow xz = 2\lambda(x+z) \\
f_z = \lambda G_z &\Rightarrow xy = 2\lambda(x+y) \\
xyz &= 2\lambda(xy+xz) \\
xyz &= 2\lambda(xy+yz) \\
xyz &= 2\lambda(xz+yz) \\
\frac{xyz}{2\lambda} &= xy + xz = xy + yz = xz + yz \\
x = y = z &= \sqrt{\frac{A}{6}}
\end{aligned}$$

$$V = \left(\frac{A}{6}\right)^{\frac{3}{2}}$$

Suppose $x \leq y \leq z$

$$2x^2 \leq 2xy \leq A \Rightarrow x \leq \sqrt{\frac{A}{2}}$$
$$2yz \leq 2xz \leq A \Rightarrow yz \leq \frac{A}{2}$$
$$V = xyz \leq \sqrt{\frac{A}{2}} \cdot \frac{A}{2}$$

Volume is bounded, at the max $\nabla f = \lambda \neq G$. So $V = \left(\frac{A}{6}\right)^{\frac{3}{2}}$ is the maximum.

Ex 2.19. *Find the minimum surface area of a box that has volume V.*

$$V = xyz$$
$$G(x, y, z) = xyz - V = 0$$
$$A = f(x, y, z) = 2xy + 2xz + 2yz$$

area is not bounded above

$$f_x = 2(y + z) = \lambda yz$$
$$f_y = 2(x + z) = \lambda xz$$
$$f_z = 2(x + y) = \lambda xy$$
$$x = y = z = \sqrt[3]{\frac{V}{3}}$$

2.6 $\mathbb{R}^n \to \mathbb{R}^m$

Definition 2.10. *$f : \mathbb{R}^n \to \mathbb{R}^m$, f is differentiable at $\vec{a}$ means there is a matrix A representing a linear transformation so that equations*

$$f(\vec{a} + \vec{h}) = f(\vec{a}) + A\vec{h} + o\left(\left|\vec{h}\right|\right)$$

call $Df(\vec{a}) = A$

Remark 2.3. *$f : \mathbb{R}^n \to \mathbb{R}^m$, $f(\vec{x}) = (f_1(x), \ldots, f_m(\vec{x}))$ then*

$$D(\vec{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} = \left(\frac{\partial f_i}{\partial x_j}\right) \qquad \begin{matrix} i = 1, \cdots, m & \textit{rows} \\ j = 1, \cdots, n & \textit{cols} \end{matrix}$$

Remark 2.4. *$f(\vec{x}) = A\vec{x} + \vec{b}$, $Df(\vec{x}) = A$.*

Proof. Method 1.

$$\begin{aligned} f(\vec{x} + \vec{h}) &= A(\vec{x} + \vec{h}) + \vec{b} \\ &= A\vec{x} + A\vec{h} + \vec{b} \\ &= A\vec{x} + \vec{b} + A\vec{h} \\ f(\vec{x} + \vec{h}) &= f(\vec{x}) + A\vec{h} \\ \Rightarrow Df &= A \end{aligned}$$

Method 2.

$$
\begin{aligned}
f_1(\vec{x}) &= \sum_{l=1}^{n} a_{1l}x_l + b_1 \\
f_2(\vec{x}) &= \sum_{l=1}^{n} a_{2l}x_l + b_2 \\
&\vdots \\
f_i(\vec{x}) &= \sum_{l=1}^{n} a_{il}x_l + b_i \\
\frac{\partial f_i}{\partial x_j} &= a_{ij} \\
Df &= \left(\frac{\partial f_i}{\partial x_j} = A\right)
\end{aligned}
$$

□

Theorem 2.20 (Chain Rule)**.** $Dg \circ f = Dg(f)Df$

Ex 2.20. *Between polar and rectangular coordinate,* $(x, y) = f(r, \theta)$*.*

$$
Df(r,\theta) = \begin{bmatrix} x_r & x_\theta \\ y_r & y_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}
$$

Ex 2.21. $g(x, y) = (u, v), u = x^2 - y^2, v = 2xy$*.*

$$
\begin{aligned}
Dg(x,y) &= \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix} \\
Dg \circ f &= Dg(f)Df \\
&= \begin{bmatrix} 2r\cos\theta & -2r\sin\theta \\ 2r\sin\theta & 2r\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \\
&= \begin{bmatrix} 2r(\cos^2\theta - \sin^2\theta) & -4r^2\sin\theta\sin\theta \\ 4r\cos\theta\sin\theta & 2r^2\cos(\cos^2\theta - \sin^2\theta) \end{bmatrix}
\end{aligned}
$$

Ex 2.22. $f : \mathbb{R}^3 \to \mathbb{R}^2, (x, y, z) \in \mathbb{R}^3, (u, v) \in \mathbb{R}^2, f(x, y, z) = (u, v) = (2xy^2\sin(z), 3xe^{2y-5z})$.

$$
\begin{aligned}
Df &= \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \end{bmatrix} = \begin{bmatrix} 2y^2\sin(z) & 4xy\sin(z) & 2xy^2\cos(z) \\ 3e^{2y-5z} & 6xe^{2y-5z} & -15xe^{2y-5z} \end{bmatrix} \\
\frac{\partial(u,v)}{\partial(x,y)} &= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \\
&= u_x v_y - u_y v_x \\
&= 12xy^2\sin(z)e^{2y-5z} - 12xy\sin(z)e^{2y-5z} \\
\frac{\partial(u,v)}{\partial(y,z)} &= -60x^2y^2\sin(z)e^{2y-5z} - 12x^2y^2\cos(z)e^{2y-5t}
\end{aligned}
$$

Ex 2.23. $f : \mathbb{R}^3 \to \mathbb{R}^3, z = \rho\cos\phi, y = \rho\sin\phi\sin\theta, x = \rho\sin\phi\cos\theta$

$$
\begin{aligned}
Df &= \begin{bmatrix} x_\rho & x_\phi & x_\theta \\ y_\rho & y_\phi & y_\theta \\ z_\rho & z_\phi & z_\theta \end{bmatrix} \\
&= \begin{bmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{bmatrix}
\end{aligned}
$$

Jacobian of f*, or* $\det Df = J_f =$

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} &= \cos\phi\cdot(\rho^2\cos^2\theta\cos\phi\sin\phi+\rho^2\sin^2\theta\sin\phi\cos\phi) \\
&\qquad +\rho\sin\phi(\rho\sin^2\phi\cos^2\theta+\rho\sin^2\phi\cos^2\theta) \\
&= \cos\phi\rho^2\cos\phi\sin\phi+\rho^2\sin^3\phi \\
&= \rho^2\sin\phi
\end{aligned}$$

Chapter 3

Implicit Function Theorem and Its Applications

3.1 Implicit Function Theorem

Theorem 3.1 (IVF, Version One). $F(\vec{x}, y), \vec{x} \in \mathbb{R}^n, y \in \mathbb{R}, F \in C^1(U), U$ *open in* $\mathbb{R}^{n+1}.\exists(\vec{a}, b) \in U.$ $F(\vec{a}, b) = 0, F_y(\vec{a}, b) \neq 0$*, there are balls* $B(\vec{a}, r_0), B(b, r_1)$ *so that for each* $\vec{x} \in B(\vec{a}, r_0)$*, there is a unique* $y \in B(b, r_1)$*, we call* $y = f(\vec{x})$*, then* $f \in C^1(B(\vec{a}, r_0))$ *and*

$$\frac{\partial f(\vec{x})}{\partial x_j} = \frac{-F_{x_j}(\vec{x}, f(\vec{x}))}{F_y(\vec{x}, f(\vec{x}))}$$

Proof. Take $B(\vec{a}, \hat{r_0}), B(b, r_1)$, without lost of generosity, assume $F_y(\vec{a}, b) > 0$ in $B(\vec{a}, \hat{r_0}) \times B(b, r_1)$; then F is positive in the neighborhood. Then there's subset at intersection of direction of y and the neighborhood boundary being positive with length $\hat{r_0}^+$ and a subset at intersection of opposite direction of y and the neighborhood boundary being negative with length $\hat{r_0}^-$. Let $r_0 = \min(\hat{r_0}^+, \hat{r_0}^-)$. By MVT, for all $\vec{x} \in B(\vec{a}, r_0)$ there is a unique $y \in B(b, r_1)$ with $F(\vec{x}, y) = 0$, call $y = f(\vec{x})$. This means $F(\vec{x}, f(\vec{x})) = 0$ for all $\vec{x} \in B(\vec{a}, \vec{r_0})$.

For $\frac{\partial f}{\partial x_j}$, let $\vec{h} = h\vec{e_j}$,

$$F(\vec{x} + \vec{h}, f(\vec{x} + \vec{h})) - F(\vec{x}, f(\vec{x})) = 0$$
$$F(\vec{x} + \vec{h}, f(\vec{x} + \vec{h})) - F(\vec{x}, f(\vec{x} + \vec{h})) + F(\vec{x}, f(\vec{x} + \vec{j})) - F(\vec{x}, f(\vec{x})) = 0$$

By MVT, let $\left|\vec{t}\right| \leq \left|\vec{h}\right|$, s between $f(\vec{x} + \vec{h}), f(\vec{x})$

$$F_{x_j}(\vec{x} + \vec{t}, f(\vec{x} + \vec{h}))(h - 0) + F_y(\vec{x}, \vec{s})(f(\vec{x} + \vec{h}) - f(\vec{x})) = 0$$
$$\frac{f(\vec{x} + \vec{h}) - f(\vec{x})}{h} = \frac{-F_{x_j}(\vec{x} + \vec{t}, f(\vec{x} + \vec{h}))}{F_y(\vec{x}, s)}$$

Let $h \to 0, \vec{t} \to 0, s \to f(\vec{x})$

$$\lim_{h \to 0} \frac{f(\vec{x} + \vec{h}) - f(\vec{x})}{h} = \frac{-F_{x_j}(\vec{x}, f(\vec{x}))}{F_y(\vec{x}, f(\vec{x}))}$$

□

Remark 3.1 (Application)**.** *If*

- $F(x_1, \ldots, x_n, y) = 0$
- $F(\vec{a}, b) = 0$
- $F_y(\vec{a}, b) \neq 0$
- F *is* C^1 *near* $(\vec{a}, b)$

then $y = f(\vec{x})$ *for* $\vec{x}$ *near* $\vec{a}$*,* $F(\vec{x}, f(\vec{x})) = 0$*.*

Theorem 3.2. $S = \{\vec{x} \in \mathbb{R}^n \mid G(\vec{x}) = 0\}, \vec{a} \in S \iff G(\vec{a}) = 0$*. If* $\nabla G(\vec{a}) \neq \vec{0}$*,*

$$(\partial_{x_1} G(\vec{a}), \partial_{x_2} G(\vec{a}), \ldots, \partial_{x_n} G(\vec{a})) \neq \vec{0}$$

then for some k*,* $\partial_{x_k} G(\vec{a}) \neq 0$*. By the IFT,* $x_k = f(x_1, \ldots, \underbrace{\hat{x_k}}_{\text{removed}}, \ldots, \vec{x}_n)$ *near* $\vec{a}$*. There is a neighborhood* N *of* $\vec{a}$ *os that* $N \cap S$ *is a graph. If* $\nabla G \neq 0$ *on al of* S *then* S *is locally a graph at each point.*

Ex 3.1. *Solve* $x^2 - 4x + 2y^2 - yz - 1 = 0$ *for* $x(y, z), y(x, z), z(x, y)$ *near* $(2, -1, 3)$*.*

Answer.

$$\begin{aligned} F(x, y, z) &= x^2 - 4x + 2y^2 - yz - 1 \in C^1 \\ F(2, -1, 3) &= 0 \end{aligned}$$

For x as a function of y and z, check $F_x(2, -1, 3) \neq 0$

$$F_x(2, -1, 3) = 2x - 4|_{x=2} = 0 \qquad \text{(IFT does not apply)}$$

For z as a function of x and y, check $F_z(2, -1, 3) \neq 0$

$$\begin{aligned} F_z(2, -1, 3) &= -y|_{y=-1} = 1 \qquad \text{(IFT does apply)} \\ \frac{\partial z}{\partial x} &= \frac{-F_x}{F_z} = \frac{-(2x-4)}{-y} \\ \frac{\partial z}{\partial y} &= \frac{-F_x}{F_y} = \frac{-(4y-z)}{-y} \end{aligned}$$

Theorem 3.3 (IVT, In General)**.**

$$\begin{gathered} \vec{F}(\vec{x}, \vec{y}) = 0 \\ \vec{y} = (y_1, \ldots, y_m) \\ \vec{x} = (x_1 \ldots, x_n) \\ \vec{F}(\vec{a}, \vec{b}) = 0, \vec{F} \in C^1 \textit{ near } (\vec{a}, \vec{b}) \\ \det D_{\vec{y}} \vec{F} = \left. \frac{\partial(F_1, \ldots, F_m)}{\partial(y_1, \ldots, y_m)} \right|_{(\vec{a}, \vec{b})} \neq 0 \\ \vec{y}(\vec{x}) \textit{ for } \vec{x} \textit{ near } \vec{a} \\ F(\vec{x}, \vec{y}(\vec{x})) = 0, \vec{y} \in C^1 \end{gathered}$$

Then

$$\frac{\partial y_i}{\partial x_j} = \frac{\frac{\partial(F_1, \ldots, F_m)}{\partial(y_1, \ldots, y_m)}}{\frac{\partial(F_1, \ldots, F_m)}{\partial(y_1, \ldots, y_m)}} \qquad \text{(Except } y_i \text{ is replaced by } x_j \text{ in numerator)}$$

Ex 3.2.

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases} \Rightarrow \begin{cases} F = \rho\sin\phi\cos\theta - x = 0 \\ G = \rho\sin\phi\sin\theta - y = 0 \\ H = \rho\cos\phi - z = 0 \end{cases}$$

Solve for ρ, ϕ, θ as functions of x, y, t?

Answer. $\frac{\partial(F,G,H)}{\partial(\rho,\phi,\theta)} = \rho^2\sin\phi \neq 0$ for $\rho > 0, 0 < \phi < \pi$. So yes, ρ, ϕ, θ are functions of x, y, z.

$$\begin{aligned} \frac{\partial\phi}{\partial y} &= \frac{-\frac{\partial(F,G,H)}{\partial(\rho,y,\theta)}}{\frac{\partial(F,G,H)}{\partial(\rho,\phi,\theta)}} \\ &= \frac{-\frac{\partial(F,G,H)}{\partial(\rho,y,\theta)}}{\rho^2\sin\phi} \\ \text{where} \quad \frac{\partial(F,G,H)}{\partial(\rho,y,\theta)} &= \begin{vmatrix} F_\rho & F_y & F_\theta \\ G_\rho & G_y & G_\theta \\ H_\rho & H_y & H_\theta \end{vmatrix} = \begin{vmatrix} F_\rho & 0 & F_\theta \\ \dots & -1 & \dots \\ H_\rho & 0 & H_\theta \end{vmatrix} \\ &= -(F_\rho H_\theta - F_\theta H_\rho) \\ \text{Notice that } H_\theta = 0, \quad &= F_\theta H_\rho \\ &= -\rho\sin\phi\sin\theta\cos\phi \end{aligned}$$

3.2 Curve in $\mathbb{R}^2$

Definition 3.1 (Smooth Curve)**.**

1. *Graph of a C^1 function on an interval, $y = f(x)$ or $x = g(y)$.*
2. ***Locus.** $S = \{(x,y) \mid F(x,y) = 0\}$. If $\nabla F(a,b) \neq 0$ forall $(a,b) \in S$, then either*
   - *(a) $F_x(a,b) \neq 0$, meaning $x(y)$ for y near b, or*
   - *(b) $F_y(a,b) \neq 0$, meaning $y(x)$ for x near b*

   $\Rightarrow$ S is locally a connected C^1 graph.
3. ***Parametric.** $\vec{f} : (a,b) \to \mathbb{R}^2$, $\vec{f}$ is C^1, so connected.*

$$\begin{aligned} \vec{f}(t) &= (x(t), y(t)) \\ \vec{f}'(t) &= (x'(t), y'(t)) \end{aligned}$$

   check that f is 1-1.

Ex 3.3. $F(x,y) = (x^2+y^2-1)(x^2+y^2-4) = 0$. *Is $S = \{(x,y) \mid F(x,y) = 0\}$ a smooth curve?*
Answer. *$x^2+y^2-1 = 0$ or $x^2+y^2-4 = 0$, not connected, so not a smooth curve.*

Ex 3.4. *Is $S = \left\{(x,y) \mid x^2 = y^2\right\}$ a smooth curve?*

$$\begin{aligned} F(x,y) &= x^2 - y^2 = 0 \\ \nabla F &= \begin{pmatrix} 2x \\ -2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{for } (x,y) = \vec{0} \end{aligned}$$

It's not a graph for either x or y near $\vec{0}$. So not a smooth curve.

Ex 3.5. *$\vec{f}(t) = (\cos t, \sin t), 0 \le t < 2\pi$, the counter-clockwise unit circle, is a smooth curve. $\vec{g}(t) = (\cos t, \sin t), 0 \le t \le 4\pi$, not 1-1 so not a smooth curve.*

Ex 3.6. *Is $\vec{f}(t) = (t^2 - 1, t^2 + 1)$ for $t \in \mathbb{R}$ a smooth curve?*
Answer. *$x = t^2 - 1, y = t^2 + 1, \vec{f}(t)$ is on the line $y = x + 2$. $\vec{f}(-1) = f(1) = (0, 2)$. So not smooth.*

Ex 3.7. *$S = \{(x, y) \mid F(x, y) = x^2 - 3y^2 - 3 = 0\}$. Is S a smooth curve?*
Answer.

$$\nabla F(x,y) = \begin{pmatrix} 2x \\ -6y \end{pmatrix} = \vec{0} \quad \textit{for } (x,y) = \vec{0} \notin S$$
$$F \in C^1$$

S is locally a graph at every point in S. But it's a hyperbola, so not connected. Not a smooth curve.

3.3 Smooth Surfaces

1. Graphs (all functions are C')

$$z = f(x,y) \text{ or } y = g(x,z) \text{ or } x = h(y,z)$$
$$f(x,y) - z = 0$$
$$F(x,y,z) = 0$$

   $\nabla F(x, y, z)$ is a normal to the surface

$$\vec{n} = \nabla F = (F_x, F_y, F_z) = (f_x, f_y, -1)$$

2. Locus $S = \{(x, y, z) \mid F(x, y, z) = 0\}$. $\nabla F(x, y, z) \neq 0$ for all $(x, y, z) \in S$. If $F_x \neq 0$ then $x = h(y, z)$ near the point (x, y, z). So S is a graph $h(y, z), y, z$ near the point. Similar for $F_y \neq 0$ and $F_z \neq 0$.

3. Parametric (want 1-1)

$$\vec{f}(u,v) = (x(u,v), y(u,v), z(u,v))$$

   Fix $v = v_0$

$$\vec{f}(u) = (x(u,v_0), y(u,v_0), z(u,v_0))$$
$$\vec{g}(u) = (x(u_0,v), y(u_0,v), z(u_0,v))$$
$$\vec{f}_u = (x_u, y_u, z_u)$$
$$\vec{g}_v = (x_v, y_v, z_v)$$

   The normal is

$$f_u \times g_v = (y_u z_v - z_u y_v, z_u x_v - x_u z_v, x_u y_v - y_u x_v) \neq 0$$

Ex 3.8 (Locus). $x^2 + y^2 + z^2 = 1$.

$$S = \left\{ \underbrace{x^2 + y^2 + z^2}_{F(x,y,z)} - 1 = 0 \right\}$$
$$\nabla F = (2x, 2y, 2z) = 2(x, y, z) = 0 \textit{ only at } (0,0,0) \notin S$$

Find graph for top half

$$
\begin{gathered}
= \sqrt{1-x^2-y^2} \\
(x, y, \sqrt{1-x^2-y^2}) \quad x^2+y^2 \leq 1
\end{gathered}
$$

Find graph for front half

$$
(\sqrt{1-y^2-z^2}, y, z) \quad y^2+z^2 \leq 1
$$

Ex 3.9 (Parametric)**.** *Sphere of radius 1. Use spherical coordinates:*

$$
\begin{gathered}
x = \sin\phi\cos\theta \\
y = \sin\phi\sin\theta \\
z = \cos\phi \\
\textit{The surface is } (x(\phi,\theta), y(\phi,\theta), z(\phi,\theta)) \\
(\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi) \times (-\sin\phi\sin\theta, \sin\phi\cos\theta, 0) \\
= (\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \cos\phi\sin\phi) \\
= \sin\phi(\underbrace{\sin\phi\cos\theta}_{x}, \underbrace{\sin\phi\sin\theta}_{y}, \underbrace{\cos\phi}_{z}) \\
\vec{n} = 0 \textit{ for } \phi = 0, \pi
\end{gathered}
$$

Ex 3.10. $\vec{f}(u,v) = (u\cos v, u\sin v, u^2), (u,v) \in \mathbb{R}^2$. *For 1-1, must be* $0 < v < 2\pi, u > 0$.

$$
\begin{aligned}
\vec{f_u} \times \vec{f_v} &= (\cos v, \sin v, 2u) \times (-u\sin v, u\cos v, 0) \\
&= (-2u^2\cos v, -2u^2\sin v, u\cos^2 v + u\sin^2 v) \\
&= (-2u^2\cos v, -2u^2\sin v, u) \\
&= u(-2u\cos v, -2u\sin v, 1) \\
\vec{f}(-u,v) &= (-u\cos v, u\sin v, u^2) \\
&= (u(-\cos v), u(-\sin v), u^2) \\
&= (u\cos(v+\pi), u\sin(v+\pi), u^2) \\
&= f(u, v+\pi)
\end{aligned}
$$

Let $x = u\cos v, y = u\sin v, z = u^2$*,* $\vec{f}(u,v)$ *is on* $x^2+y^2 = z$ *(Locus)*

$$
\begin{aligned}
F(x,y,z) &= x^2+y^2-z = 0 \\
\nabla F &= (2x, 2y, -1) \neq 0
\end{aligned}
$$

The entire parabola is a smooth curve

3.4 Inverse Function Theorem

Ex 3.11.

$$
\begin{aligned}
x^2 \le y \le 2x^2 \to 1 \le \frac{y}{x^2} \le 2 \\
1 \le & xy \le 3 \\
\textit{let } u = & \frac{y}{x^2}, v = xy \\
Df = & \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \\
= & \begin{bmatrix} \frac{-2x}{x^3} & \frac{1}{x^2} \\ y & x \end{bmatrix} \\
J_f = & \frac{-3y}{x^2} \neq 0
\end{aligned}
$$

By IFT, f^{-1} *exists*

$$
\begin{aligned}
Df^{-1}(u,v) &= \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} \\
&= (Df(x,y))^{-1} \\
&= \frac{1}{-3y/x^2} \begin{bmatrix} x & -\frac{1}{x^2} \\ -y & -\frac{2x}{x^3} \end{bmatrix}
\end{aligned}
$$

Chapter 4

Integration

4.1 Integration in $\mathbb{R}$

Definition 4.1. *$f : [a, b] \to \mathbb{R}$, f is bounded, $|f(x)| \le M$, $\forall x \in [a, b]$. , $-M \le f(x) \le M$. Let partition $P = \{x_0 = a < x_1 < x_2 < \cdots < x_n = b\}$. Subintervals $I_k = [x_{k-1}, x_k], k = 1, \ldots, n$ with widths $|I_k| = x_k - x_{k-1} = \Delta x_k$. Let $m_k = \inf_{x \in I_k} f(x), M_k = \sup_{x \in I_k} f(x)$. Then the **lower sum** is*

$$L_p = \sum_{k=1}^{n} m_k \Delta x_k$$

*the **upper sum** is*

$$U_p = \sum_{k=1}^{n} M_k \Delta x_k$$

*and f is **integrable** if and only if*

$$\forall \epsilon > 0.\ \exists P.\ U_p - L_p < \epsilon$$

if true then

$$\int_a^b f(x)\ dx$$

Theorem 4.1. *If f is monotone on $[a, b]$ then f is integrable.*

Theorem 4.2. *If f is continuous on $[a, b]$ then f is integrable.*

Definition 4.2 (Sets with Zero Content)**.** *$S \subseteq \mathbb{R}$ has **zero content** means for every $\epsilon > 0$, there is a finite collection of intervals $I_1, \ldots, I_n$ so that*

$$S \subseteq \bigcup_{k=1}^{n} I_k$$

and

$$\sum_{k=1}^{n} |I_k| < \epsilon$$

Ex 4.1. *$S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$. Given $\epsilon > 0$, let*

$$I_0 = \left(0 - \frac{\epsilon}{4}, 0 + \frac{\epsilon}{4}\right) = B\left(0, \frac{\epsilon}{4}\right)$$
$$|I_0| = \frac{\epsilon}{2}$$

Since $\frac{1}{n} \to 0$ *as* $n \to \infty$*, so* $\exists N.\ n \geq N+1 \Rightarrow \frac{1}{n} \in I_0$*,*

$$\left\{ \frac{1}{n+1}, \frac{1}{n+2}, \dots \right\} \subseteq I_0$$

And for $1, \dots, \frac{1}{N}$*,*

$$\begin{aligned}
B\left(1, \frac{\epsilon}{4N}\right) &= I_1, \quad |I_1| = \frac{\epsilon}{2N} \\
B\left(2, \frac{\epsilon}{4N}\right) &= I_2, \quad |I_2| = \frac{\epsilon}{2N} \\
&\vdots \\
B\left(N, \frac{\epsilon}{4N}\right) &= I_N, \quad |I_N| = \frac{\epsilon}{2N} \\
S &\subseteq \bigcup_{k=0}^{N} I_k \\
\sum_{k=0}^{N} |I_l| &= \frac{\epsilon}{2} + \sum_{l=1}^{N} |I_e| \\
&= \frac{\epsilon}{2} + \sum_{l=1}^{N} \left| \frac{\epsilon}{2N} \right| \\
&= \frac{\epsilon^*}{2} + \frac{\epsilon^*}{2} \\
&= \epsilon^* < \epsilon
\end{aligned}$$

Theorem 4.3. *If* $R \subseteq S$*,* S *has zero content, then* R *has zero content.*

Theorem 4.4. *If* $S_1, S_2, \dots, S_j$ *have zero content, then*

$$S = \bigcup_{j=1}^{\infty} S_j$$

has zero content.

Theorem 4.5. *If* f *is continuous on* $[a, b]$ *except on* S *a set with zero content, then* f *is integrable.*

4.2 Integration in $\mathbb{R}^2$

Definition 4.3. f *bounded on a rectangle,* $R = [a, b] \times [c, d]$*. Partition:*

$$\begin{aligned}
x_0 = a < x_1 < x_2 < \dots < x_n = b \\
y_0 = c < y_1 < y_2 < \dots < y_m = d
\end{aligned}$$

Subinervals:

$$\begin{aligned}
I_k &= [x_{k-1}, x_k] \\
J_l &= [x_{l-1}, x_l]
\end{aligned}$$

Subrectangles:

$$R_{kl} = I_k \times J_l$$

On each R_{kl}:

$$m_{kl} = \inf_{R_{kl}} f \quad M_{kl} = \sup_{R_{kl}} f$$

Upper sum:

$$U = \sum_{j=1}^{m} \sum_{i=1}^{n} M_{kl} \underbrace{\Delta x_i \Delta y_j}_{\text{Area } R_{ij}}$$

Lower sum:

$$L = \sum_{j=1}^{m} \sum_{i=1}^{n} m_{kl} \underbrace{\Delta x_i \Delta y_j}_{\text{Area } R_{ij}}$$

$$f \text{ is integrable on } \mathbb{R} \iff \forall \epsilon > 0.\ \exists P.\ U - L < \epsilon$$

Theorem 4.6. *If f is continuous on R, then f is integrable on R.*

Definition 4.4 (Zero Content on $\mathbb{R}^2$). *$S \subseteq \mathbb{R}^2$ has zero content means for every $\epsilon > 0$, there is a finite collection of rectangles $R_1, \ldots, R_N$ so that*

$$S \subseteq \bigcup_{i=1}^{N} R_i \quad \text{and} \quad \sum_{i=1}^{N} |R_i| < \epsilon$$

Theorem 4.7. *If g is integrable on $[a, b]$, then $S = \{(x, g(x)) \mid a \leq x \leq b\}$ has zero content in $\mathbb{R}^2$*

Proof. Since g is integrable, given $\epsilon > 0$, there is a partition on so that

$$U - L = \sum_{i=1}^{n} (M_i - m_i) \Delta x_i < \epsilon$$

then

$$\begin{gathered} R_i = I_i \times [m_i, M_i] \subseteq S \\ |R_i| = \Delta x_i (M_i - m_i) \\ \text{on } I_i, m_i \leq g(x) \leq M_i \\ \text{so } S \subseteq \bigcup_{i=1}^{n} R_i, \quad \sum_{i=1}^{n} |R_i| < \epsilon \end{gathered}$$

□

Theorem 4.8. *A vertical line segment $\{x\} \times [c, d]$ has zero content in $\mathbb{R}^2$.*

Theorem 4.9. *If $S_1, S_2, \ldots, S_N$ have zero content in $\mathbb{R}^2$ then*

$$S = \bigcup_{i=1}^{N} S_i$$

has zero content in $\mathbb{R}^2$.

Theorem 4.10. *If f is continuous on a rectangle $R \subseteq \mathbb{R}^2$ except on a set S of zero content, then f is integrable.*

4.3 Evaluate Integrals

Definition 4.5 (Multiple Integral). *$R = [a,b] \times [c,d]$, by Fubini-Tonelli,*

$$\iint_R f(x,y)\;dx\;dy = \int_c^d \left(\int_a^b f(x,y)\;dx \right)\;dy = \int_a^b \left(\int_c^d f(x,y)\;dy \right)\;dx$$

Ex 4.2. *S is the region bounded by $y = x^2, y = 6 - 4x - x^2$. Write $\iint_S f dA$ as iterated integrals.*

$$\begin{aligned}
y &= -(x^2 + 4x - 6) \\
&= -(x+2)^2 + 10 \\
x &= \pm\sqrt{10-y} - 2 \\
x^2 &= 6 - 4x - x^2 \\
x &= 1, -3
\end{aligned}$$

$$\begin{aligned}
\int_S f dA &= \int_{-3}^1 \int_{x^2}^{-(x+2)^2+10} f(x,y)\;dy\;dx \\
&= \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\;dx\;dy + \int_1^9 \int_{-\sqrt{y}}^{\sqrt{10-y}-2} f(x,y)\;dx\;dy + \int_9^{10} \int_{-\sqrt{10-y}-2}^{\sqrt{10-y}-2} f(x,y)\;dx\;dy
\end{aligned}$$

Ex 4.3. *S is the region bounded by $z = x^2 + y^2$ and $z = 1$.*

$$\begin{aligned}
&\iiint_S f\;dV \\
&= \int \int \int f(x,y,z)\;dx\;dy\;dz \\
&= \int \int \int f(x,y,z)\;dy\;dx\;dz \\
&= \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{x^2+y^2}^1 f(x,y,z)\;dz\;dx\;dy
\end{aligned}$$

Chapter 5

Line and Surface Integral

5.1 Curves

$$
\begin{aligned}
dR &= C_1 + C_2 + C_3 + C_4 + C_5 \\
C_1 &: (\cos t, \sin t), t \in \left[\frac{3\pi}{2}, 2\pi\right] \\
C_2 &: (1, t), t \in [0, 1] \\
C_3 &: (1 - t, 1), t \in [0, 1]
\end{aligned}
$$

In general, if we have $[a, b]$ and $\vec{g}$ from $[a, b]$ to curve c, the opposite is $-c$, then

$$
\begin{aligned}
(-\vec{g})(t) &= \vec{g}(a + b - t) \\
(-\vec{g})(a) &= \vec{g}(b) \\
(-\vec{g})(b) &= \vec{g}(a) \\
C_4 &: (1 - t, 1 - (1 + t)^2), t \in [0, 1] \\
C_5 &: (t - 1, -t)
\end{aligned}
$$

Definition 5.1 (Arclength)**.**

$$
\begin{aligned}
\vec{g} &= (x_1(t), \ldots, x_n(t)) \\
\vec{g}'(t) &= (x_1'(t), \ldots, x_n'(t)) && \text{(Velocity)} \\
\vec{g}'(t) &\neq 0, \quad \vec{g} \in C^1
\end{aligned}
$$

$$
\textit{Length} = \int \overbrace{\underbrace{|\vec{g}(t)|}_{\textit{speed}} \cdot \underbrace{dt}_{\textit{time}}}^{ds}
$$

Ex 5.1. *What is the length of c, top half of the circle with radius R?*

$$
\begin{aligned}
\vec{g}(t) &= (R \cos t, R \sin t), 0 \leq t \leq \pi \\
\vec{g}'(t) &= (-R \sin t, R \cos t) \\
|\vec{g}'(t)| &= \sqrt{R^2 \sin^2 t + R^2 \cos^2 t} = R \\
\int_C ds &= \int_0^\pi R \, dt \\
&= R\pi
\end{aligned}
$$

Ex 5.2. *Find the length of the graph of $y = x^3$ for $0 \le x \le 1$*

$$\begin{aligned}
\vec{g}(x) &= (x, x^3) \\
\vec{g}'(x) &= (1, 3x^2) \\
|\vec{g}'(x)| &= \sqrt{1 + 9x^4} \\
\int_C ds &= \int_0^1 \sqrt{1 + 9x^4}\ dx
\end{aligned}$$

Definition 5.2 (Vector Fields)**.** *For each $\vec{x} \in \mathbb{R}^n$, we have $\vec{F}(\vec{x}) \in \mathbb{R}^n$*

Definition 5.3 (Simple Closed Curve)**.** *C is a **simple closed curve** if*

1. *C is a closed curve, $\vec{g}(a) = \vec{g}(b)$*
2. *$\vec{g}$ is 1-1 on (a, b)*

Definition 5.4 (Line Integral)**.**

$$\begin{aligned}
\vec{F} &= (F_1, \dots, F_n) \\
\vec{x} = \vec{g}(t) &= (x_1, \dots, x_n), t \in [a, b] \\
\int_C \vec{F} \cdot d\vec{x} &= \int_C F_1\ dx_1 + \dots + F_n\ dx_n \\
&= \int_a^b \vec{F}(\vec{g}(t)) \cdot \vec{g}'(t)\ dt
\end{aligned}$$

Let $R \subseteq \mathbb{R}^n$

$$\begin{aligned}
C &= \partial R \\
C &: \vec{g}(t), t \in [a, b] \\
\vec{g}(a) &= \vec{g}(b) \\
T &= \frac{\vec{g}(t)}{|\vec{g}(t)|} && \text{(unit tangent vector)} \\
\int_C \vec{F} \cdot d\vec{x} &= \int_a^b \vec{F}(\vec{g}(t)) \cdot T(t)\, |\vec{g}'(t)|\ \ dt \\
&= \int_C \vec{F} \cdot T\ ds
\end{aligned}$$

If $R \in \mathbb{R}^2$, there is outward normal vector n orthogonal to T

$$\begin{aligned}
T &= (T_1, T_2) \\
n &= (T_2, -T_1) \\
\vec{F} &= (P, Q) \\
\int_C \vec{F} \cdot d\vec{x} &= \int_C \vec{F} \cdot T\ ds \\
&= \int_C PT_1 + QT_2 ds \\
&= \int_C (Q, -P) \cdot (T_2, -T_1) ds
\end{aligned}$$

Let $\vec{G} = (Q, -P)$

$$= \int_C \vec{G} \cdot n\ ds$$

Ex 5.3. *C is the unit circle, counter clockwise.*

$$\begin{aligned}
\vec{F}(x,y) &= (\underbrace{x-y}_{P}, \underbrace{x+y}_{Q}) \\
C &= (\cos t, \sin t), t \in [0, 2\pi] \\
\int_C \vec{F} \cdot d\vec{x} &= \int_C P\ dx + Q\ dy \\
&= \int_0^{2\pi} \vec{F}(\vec{g}(t)) \cdot \vec{g}'(t) dt \\
&= \int_0^{2\pi} (\cos t - \sin t, \cos t + \sin t) \cdot (-\sin t, \cos t) dt \\
&= \int_0^{2\pi} 1 dt \\
&= 2\pi
\end{aligned}$$

Ex 5.4. *C is the graph of $y = x^4$ from -2 to 2 and the line from $(2,16)$ to $(-2,16)$.*

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{x} &= \int_{C_1} \vec{F} \cdot d\vec{x} + \int_{C_2} \vec{F} \cdot d\vec{x} \\
C_1 : \vec{g}(x) &= (x, x^4), x \in [-2, 2] \\
g'(x) &= (1, 4x^3) \\
\int_{C_1} \vec{F} \cdot d\vec{x} &= \int_{-2}^{2} (x^6, x^{11}) \cdot (1, 4x^3)\ dx \\
&= \int_{-2}^{2} x^6 + 4x^{14}\ dx \\
&= 2\int_0^2 x^6 + 4x^{14}\ dx \\
&= 2\left(\frac{2^7}{7} + \frac{4 \cdot 2^{15}}{15}\right) \\
-C_2 : \vec{g}(x) &= (x, 16) x \in [-2, 2] \\
-\int_{C_2} \vec{F} \cdot d\vec{x} &= \int_{-C_2} \vec{F} \cdot d\vec{x} \\
&= \int_2^{-2} (16, x^4) \cdot (1, 0)\ dx \\
&= \int_2^{-2} 16x^2\ dx \\
&= 2 \cdot 16 \cdot \frac{2^3}{3} \\
\int_C \vec{F} \cdot d\vec{x} &= 2\left(\frac{2^7}{7} + \frac{4 \cdot 2^{15}}{15}\right) - 2 \cdot 16 \cdot \frac{2^3}{3}
\end{aligned}$$

5.2 Surface Integrals in $\mathbb{R}^3$

$$\begin{aligned} R \in \mathbb{R}^2 &= [a,b]\times[c,d] \\ \vec{G} &: R \to \mathbb{R}^3 \\ \vec{G} &: (u,v) \mapsto (x(u,v), y(u,v), z(u,v)) \\ S &= \vec{G}(R) \end{aligned}$$

Consider a small rectangle on S, the area is

$$\begin{aligned} &\left|\vec{G}_u \times \vec{G}_v\right| du\, dv \\ \vec{G}_u &= (x_u, y_u, z_u) \\ \vec{G}_v &= (x_v, y_v, z_v) \\ \text{Area of } S &= \iint_S dA \\ &= \iint_R \left|\vec{G}_u \times \vec{G}_v\right| du\, dv \\ \vec{G}_u \times \vec{G}_v &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} \\ &= (y_u z_v - z_u y_v, z_u x_v - x_u z_v, x_u y_v - y_u x_v) \\ &= \left(\frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)}\right) \left|\vec{G}_u \times \vec{G}_v\right| \\ &= \sqrt{\left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2} \end{aligned}$$

Find $\iint_S \vec{F}\cdot n dA$ where $\vec{F} \in C^1$ is a vector field

$$\begin{aligned} \iint_S \vec{F}\cdot n dA &= \iint_R \vec{F}(\vec{G}) \cdot n \left|\vec{G}_u \times \vec{G}_v\right|\ du\, dv \\ &= \iint_R \vec{F}(\vec{G}) \cdot \frac{\vec{G}_u \times \vec{G}_v}{\left|\vec{G}_u \times \vec{G}_v\right|} \left|\vec{G}_u \times \vec{G}_v\right|\ du\, dv \\ &= \iint_R \vec{F}(\vec{G}) \cdot (\vec{G}_u \times \vec{G}_v)\ du\, dv \end{aligned}$$

Ex 5.5. *Area of the upper hemisphere of radius a.*

Method 1

$$\begin{aligned} x^2 + y^2 + z^2 &= a^2 \\ z &= \sqrt{a^2 - x^2 - y^2} && \text{(Graph over } x^2 + y^2 \le a\text{)} \\ \vec{G}(x,y) &= (x, y, \sqrt{a^2 - x^2 - y^2}) \\ \vec{G}_x &= \left(1, 0, -\frac{x}{z}\right) \\ \vec{G}_x &= \left(0, 1, -\frac{y}{z}\right) \\ \vec{G}_x \times \vec{G}_y &= \left(\frac{x}{z}, \frac{y}{z}, 1\right) && \text{(Normal to the surface, points up)} \end{aligned}$$

$$
\begin{aligned}
n &= \frac{\left(\frac{x}{z}, \frac{y}{z}, 1\right)}{\sqrt{\frac{x^2}{z^2} + \frac{y^2}{z^2} + 1}} \\
dA &= \sqrt{\frac{x^2}{z^2} + \frac{y^2}{z^2} + 1}\ dx\ dy \\
&= \sqrt{\frac{x^2 + y^2 + z^2}{z^2}}\ dx\ dy \\
&= \frac{a}{z}\ dx\ dy \\
\textit{Area of } S &= \iint_R dA \\
&= \int_0^{2\pi} \int_0^a \frac{a}{\sqrt{a^2 - r^2}} r\ dr\ d\theta \\
&= 2\pi a(-\sqrt{a^2 - r^2})\Big|_0^a \\
&= 2\pi a^2
\end{aligned}
$$

Method 2

$$
\begin{aligned}
\vec{G}(\phi, \theta) &= (a \sin\phi \cos\theta, a \sin\phi \sin\theta, a \cos\phi) & (0 < \phi < \pi/2, 0 < \theta < 2\pi) \\
\vec{G}_\phi &= (a \cos\phi \cos\theta, a \cos\phi \sin\theta, -a \sin\phi) \\
&= (-a \sin\phi \sin\theta, a \sin\phi \cos\theta, 0) \\
\vec{G}_\phi \times \vec{G}_\psi &= (a^2 \sin^2\phi \cos\theta, a^2 \sin^2\phi \sin\theta, a^2 \cos\phi \sin\phi \cos^2\theta + a^2 \cos\phi \sin\phi \sin^2\theta) \\
&= a \sin\phi (a \sin\phi \cos\theta, a \sin\phi \sin\theta, a \cos\phi) \\
&= a \sin\phi \vec{G}(\phi, \theta) \\
\left|\vec{G}_\phi \times \vec{G}_\psi\right| &= a \sin\phi \left|\vec{G}(\phi, \theta)\right| \\
&= a^2 \sin\phi \\
\textit{Area of } S &= \iint_R a^2\ \sin\ d\phi\ d\theta \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} a^2 \sin\phi\ d\phi\ d\theta \\
&= 2\pi a^2 (-\cos\phi)\Big|_0^{\frac{\pi}{2}} \\
&= 2\pi a^2
\end{aligned}
$$

Ex 5.6. *Given box* $(x, y, z) \in [0, a] \times [0, b] \times [0, c]$*,* S *is the surface of the box.* $\vec{F}(x, y, z) = (x, y, z)$.

$$
\begin{aligned}
\iint_S \vec{F} \cdot n\ dA &= \cancel{\iint_{\textit{bot}} \vec{F} \cdot n\ dA} + \cancel{\iint_{\textit{left}} \vec{F} \cdot n\ dA} + \cancel{\iint_{\textit{back}} \vec{F} \cdot n\ dA} \\
&\quad + \iint_{\textit{top}} \vec{F} \cdot n\ dA + \iint_{\textit{right}} \vec{F} \cdot n\ dA + \iint_{\textit{front}} \vec{F} \cdot n\ dA \\
&= \iint_{\textit{top}} \underbrace{\vec{F} \cdot n}_{z=c}\ dA + \iint_{\textit{right}} \underbrace{\vec{F} \cdot n}_{y=b} \cdot n\ dA + \iint_{\textit{front}} \underbrace{\vec{F} \cdot n}_{x=a} \cdot n\ dA \\
&= 3abc
\end{aligned}
$$

5.3 Divergence Theorem

Theorem 5.1 (Divergence Theorem). *$R \in \mathbb{R}^3$, compact, regular ($R = \bar{R^{int}}$), ∂R piecewise smooth, oriented (outward normal is C^1 on S). $\vec{F}$ is C^1 on R. Then*

$$\begin{aligned} \iint_{\partial R} \vec{F} \cdot n \, dA &= \iiint_R div \cdot \vec{F} \, dV \\ div(\vec{F}) &= \nabla \cdot \vec{F} \\ &= \partial_x F_1 + \partial_y F_2 + \partial_z F_3 \end{aligned}$$

Ex 5.7. *Cylinder, with radius a and from $z \in [0,2]$. $\partial R =$ Top + Bottom + Side. $\vec{F} = (x^2 + y^2, -2xy, z^3 + xy)$*

$$\begin{aligned} \iint_{\partial R} \vec{F} \cdot \vec{n} \, dA &= \iiint_R div \, \vec{F} \, dV \\ &= \iiint_R 2x - 2x + 3z^2 \, dx \, dy \, dz \\ &= \iiint_R 3z^2 \, dx \, dy \, dz \\ &= \iint_{x^2+y^2 \le a^2} \int_0^2 3z^2 \, dz \, dx \, dy \\ &= \iint_{x^2+y^2 \le a^2} 8 \, dx \, dy \\ &= 8\pi a^2 \end{aligned}$$

Ex 5.8 (Fundamental Solution). *In $\mathbb{R}^3$, $\vec{x} = (x, y, z)$,*

$$\begin{aligned} g(\vec{x}) &= \frac{1}{|\vec{x}|} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \\ \partial_x g &= -\frac{x}{|\vec{x}|^3} \\ \partial_y g &= -\frac{y}{|\vec{x}|^3} \\ \partial_z g &= -\frac{z}{|\vec{x}|^3} \\ \nabla g &= -\frac{\vec{x}}{|\vec{x}|^3} \\ \Delta g = \nabla \cdot \nabla g &= \partial_x g_x + \partial_y g_y + \partial_z g_z \\ &= \partial_x \frac{-x}{|\vec{x}|^3} + \partial_y \frac{-y}{|\vec{x}|^3} + \partial_z \frac{-z}{|\vec{x}|^3} \\ &= \frac{-3}{|\vec{x}|^3} - x\left(-3\,|\vec{x}|^{-5}\, x\right) - y\left(-3\,|\vec{x}|^{-5}\, y\right) - z\left(-3\,|\vec{x}|^{-5}\, z\right) \\ &= \frac{-3}{|\vec{x}|^3} + \frac{3}{|\vec{x}|^3} = 0 \end{aligned}$$

g is harmonic in $\mathbb{R}^3$ except when $\vec{x} = 0$.

Ex 5.9. *Consider the ball with radius a. $\frac{\partial g}{\partial n}$ is the directional derivative.*

$$\begin{aligned}
\vec{n} &= \frac{1}{a}(x, y, z) \\
&= \frac{\vec{x}}{a} \\
\iint \frac{\partial g}{\partial n}\, dA &= \iint_{|\vec{x}|=a} -\frac{1}{|\vec{x}|^3}\vec{x} \cdot \frac{\vec{x}}{a}\, dA \\
&= \iint_{|\vec{x}|=a} -\frac{1}{a^2}\, dA \\
&= -\frac{1}{a^2}\iint_{|\vec{x}|=a} dA \\
&= -4\pi \neq 0
\end{aligned}$$

If we could apply the div theorem, then

$$\begin{aligned}
-4\pi &= \iint_{|\vec{x}|=a} \frac{\partial g}{\partial n}\, dA \\
&= \iiint_B \nabla \cdot \nabla g\, dV \\
&= \iiint_B 0\, dV \\
&= 0
\end{aligned}$$

So we cannot apply the div theorem.

Ex 5.10. *Suppose R is a regular region with piecewise smooth ∂R and $0 \in R^{int}$, meaning there's a ball $B(0, a) \in R$. Look $R - B(0, a)$. Note that $\partial(R - B(0, a)) = \partial R \cup \partial B(0, a)$. ∇g is a C^1 vector field on $R - B(0, a)$.*

$$\iint_{\partial R \cup \partial B(0,a)} \frac{\partial g}{\partial n}\, dA = \iiint_{R - B(0,a)} \nabla \cdot \nabla g\, dV = 0$$

Let $\hat{n}$ be the inner normal (towards origin) on the sphere

$$\begin{aligned}
\iint_{\partial R} \frac{\partial g}{\partial n}\, dA + \iint_{\partial B(0,a)} \frac{\partial g}{\partial \hat{n}}\, dA &= 0 \\
\iint_{\partial R} \frac{\partial g}{\partial n}\, dA - \iint_{\partial B(0,a)} \frac{\partial g}{\partial n}\, dA &= 0 \\
\iint_{\partial R} \frac{\partial g}{\partial n}\, dA &= -4\pi
\end{aligned}$$

Ex 5.11. *Why divergence? Consider the average value of $f(\vec{x})$ on S.*

$$\begin{aligned}
&= \frac{1}{|S|}\int_S f(\vec{x})\, d\vec{x} \\
\text{If } S &= B(x_0, r) \\
f(\vec{x}_0) &= \lim_{r \to 0} \frac{1}{|B(x_0, r)|}\int_{|B(x_0,r)|} f(\vec{x})\, d\vec{x} && \text{(when } f \text{ is cont. at } x_0\text{)}
\end{aligned}$$

Ex 5.12. *We have $\vec{F} \in C^1$ on R and $B(x_0, r) \subset R^{int}$.*

$$\begin{aligned}\operatorname{div} F(x_0) &= \lim_{r\to 0} \frac{1}{|B(x_0,r)|} \iiint_{B(x_0,r)} \operatorname{div} F(\vec{x})\, d\vec{x} \\ &= \lim_{r\to 0} \frac{1}{|B(x_0,r)|} \underbrace{\iint_{\partial B(x_0,r)} \vec{F}\cdot\vec{n}\, d\vec{x}}_{\text{the amount of stuff going out of the ball.}}\end{aligned}$$

If $\operatorname{div} F(x_0) > 0$, it's a source; if $\operatorname{div} F(x_0) < 0$, it's a sink.

5.4 Stokes's Theorem

Theorem 5.2 (Stokes's Theorem)**.** *$\vec{F} \in C^1$ is an open set containing S, S is a piecewise smooth surface with ∂S piecewise smooth, oriented (outward normal), $C = \partial S$ is a simple closed curve. Then*

$$\begin{aligned}\int_C \vec{F}\cdot T\, d\vec{x} &= \iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\, dA \\ \operatorname{curl}\vec{F} &= \nabla\times\vec{F} \\ &= (\partial_x, \partial_y, \partial_z)\times(F_1, F_2, F_3) \\ &= (\partial_y F_3 - \partial_z F_2, \partial_z F_1 - \partial_x F_3, \partial_x F_2 - \partial_y F_1)\end{aligned}$$

Why Curl? Let D_r be the disk with radius r and $C_r = \partial D_r$

$$\begin{aligned}\operatorname{curl} F(x_0)\cdot n &= \lim_{r\to 0}\frac{1}{|D_r|}\iint_{D_r}\operatorname{curl}\vec{F}\cdot n\, dA \\ &= \lim_{r\to 0}\frac{1}{|D_r|}\int_{C_r}\vec{F}\cdot T\, d\vec{x}\end{aligned}$$

If $\operatorname{curl} F(x_0)\cdot n > 0$, the vector field spend most time in the direction of tangent vectors of the circle. If $\operatorname{curl} F(x_0)\cdot n < 0$, the vector field spend most time in the opposite direction of tangent vectors. No matter what, there's curl.

Ex 5.13. $\vec{F}(x,y,z) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$ *in $\mathbb{R}^3$ except the z-axis*

$$\begin{aligned}\operatorname{curl}\vec{F} &= (\partial_y F_3 - \partial_z F_2, \partial_z F_1 - \partial_x F_3, \partial_x F_2 - \partial_y F_1) \\ &= \left(0, 0, \frac{1}{x^2+y^2} - \frac{1}{x^2+y^2}\right) \\ &= (0,0,0)\end{aligned}$$

Take $C_r = \left\{x^2+y^2 = a^2, z = 0\right\}$, then $T = \frac{1}{r}(-y, x)$

$$\begin{aligned}\int_{C_r}\vec{F}\cdot T\, ds &= \int_{C_r}\frac{-y}{x^2+y^2}\frac{-y}{r} + \frac{x}{x^2+y^2}\frac{x}{r}\, ds \\ &= \int_{C_r}\frac{r}{r^2}\, ds \\ &= 2\pi\end{aligned}$$

If there were a surface S with $\partial S = C_r$, then

$$2\pi\int_{C_r}\vec{F}\cdot T\, ds = \iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\, dA = 0$$