Real Analysis II

Lecture Notes

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Chapter 5

Integrability in $\mathbb{R}$

5.1 Riemann Integral

Definition 5.1 (Partition)**.** *$a < b$, a **partition** of $[a, b]$ is $P = \{x_0, x_1, \ldots, x_n\}$ s.t. $x_0 = a < x_1 < x_2 < \cdots < x_n = b$.*

Definition 5.2 (Norm)**.** *The **norm** of P is*

$$\|P\| = \max_{1 \leq j \leq n} |x_j - x_{j-1}|$$

Definition 5.3 (Refinement)**.** *The **refinement** of P is a partition Q such that $Q \supset P$*

Ex 5.1 (Dynamic Partition)**.** *Prove: $\forall n \in \mathbb{N}.$, $P_n = \left\{ \frac{j}{2^n} \mid 0 \leq j \leq 2^n \right\}$ is a partition of $[0, 1]$ and P_n is finer than P_n if $m > n$.*

Proof. Since $\frac{j}{2^n} < \frac{j+1}{2^n}$ and $\frac{0}{2^n} = 9$, $\frac{2^n}{2^n} = 1$, then P_n is a partition of $[0, 1]$. To show $P_m \supset P_n$:

$$\begin{aligned} \forall j \in [0, 2^n].\ \frac{j}{2^n} &= \frac{j \cdot 2^{m-n}}{2^n \cdot 2^{m-n}} \\ &= \frac{j \cdot 2^{m-n}}{2^m} \\ &\Rightarrow P_m \supset P_n \end{aligned}$$

$\square$

Remark 5.1. *If P, Q are partitions of $[a, b]$, then $P \cup Q$ is finer than P and Q.*

Remark 5.2. *Q is finer than P means $\|P\| \geq \|Q\|$.*

Definition 5.4 (Interval)**.** *For $j = 1, 2, \ldots, n$, let $\Delta x_j = x_j - x_{j-1}$ (length of j-th subinterval).*

Definition 5.5 (Upper Riemann Sum)**.** *Suppose f is a bounded function on $[a, b]$. The **Upper Riemann Sum** of f over P is*

$$U(f, P) = \sum_{j=1}^{n} M_j(f) \cdot \Delta x_j$$

where $M_j(f) = \sup_{x_{j-1} \leq x \leq x_j} f(x)$

Definition 5.6 (Lower Riemann Sum)**.** *Suppose f is a bounded function on $[a, b]$. The **Lower Riemann Sum** of f over P is*

$$L(f, P) = \sum_{j=1}^{n} m_j(f) \cdot \Delta x_j$$

where $m_j(f) = \inf_{x_{j-1} \leq x \leq x_j} f(x)$

Remark 5.3. $L(f,P) \le U(f,P)$. *Moreover, if* $f = \alpha$ *on* $[a,b]$, *then* $U(f,P) = L(f,P) = \alpha(b-a)$

Lemma 5.1. *If* $Q \supset P$, *then* $L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P)$

Proof. Without lost of generality, assume $Q = P \cup \{c\}, c \notin P$. It suffices to show that $L(f,P) \le U(f,Q)$. By definition, $P = \{x_0, \dots, x_n\}$ and $Q = \{x_0, \dots, x_l, c, x_{l+1}, \dots, x_n\}$.

$$\begin{aligned} L(f,p) &= \sum_{j=1}^{n} m_j(f)\Delta x_j \\ &= \sum_{j \neq l+1}^{n} m_j(f)\Delta x_j + m_{l+1}(f)\Delta x_{l+1} \\ &= \sum_{j \neq l+1}^{n} m_j(f)\Delta x_j + m_{l+1}(f)(x_{l+1} - x_l) \end{aligned}$$

Note that

$$\begin{aligned} \inf_{[x_l, x_{l+1}]} f &\le \inf_{[x_l, c]} f \\ \inf_{[x_l, x_{l+1}]} f &\le \inf_{[c, x_{l+1}]} f \\ m_{l+1}(f)(x_{l+1} - x_l) &= m_{l+1}(f)(x_{l+1} - c) + m_{l+1}(f)(c - x_l) \\ &\le \inf_{[c, x_{l+1}]} f(x_{l+1} - c) + \inf_{[x_l, c]} f(c - x_l) \\ L(f,Q) &= \sum_{j \neq l+1} m_j(f)\Delta x_j + \le \inf_{[c, x_{l+1}]} f(x_{l+1} - c) + \inf_{[x_l, c]} f(c - x_l) \\ &\ge \sum_{j \neq l+1} m_j(f)\Delta x_j + m_{l+1}(f)(x_{l+1} - c) + m_{l+1}(f)(c - x_l) \\ &= L(f,P) \end{aligned}$$

□

Corollary 5.1. $L(f,P) \le U(f,Q)$

Proof.

$$P \cup Q \supset P, \quad P \cup Q \supset Q$$
$$L(f,P) \le L(f, P \cup Q) \le U(f, P \cup Q) \le U(f,Q)$$

□

Definition 5.7 (Reimann Integral)**.** $f : [a,b] \to \mathbb{R}$ *is* ***Riemann Integrable*** *if*

1. f *is bounded on* $[a,b]$
2. $\forall \epsilon > 0.\ \exists P.\ U(f,P) - L(f,P) < \epsilon$

Theorem 5.1. *Every continuous functions on* $[a,b]$ *are Riemann Integrable.*

Proof. By extreme value theorem, every continuous functions on $[a,b]$ is bounded. To verify #2 in definition, firstly, if f is continuous, then f is uniformly continuous on $[a,b]$, that is,

$$\forall \epsilon > 0.\ \exists \delta > 0.\ |x_1 - x_2| < \delta \Rightarrow \left| f(x_1) - f(x_2) < \frac{\epsilon}{b-a} \right|$$

Choose partition $P = \{x_0, \dots, x_n\}$ s.t. $\|P\| < \delta$, then $|x_j - x_{j-1}| < \delta,\ j = 1, \dots, n$. Again, by extreme value theorem,

$$\exists y_j \in [x_{j-1}, x_j], z_j \in [x_{j-1}, x_j].\ f(y_j) = M_j(f), f(z_j) = m_j(f)$$

Now consider

$$\begin{aligned}
&U(f,P) - L(f,P) \\
&= \sum_{j=1}^{n} M_j(f)\Delta x_j - \sum_{j=1}^{n} m_j(f)\Delta x_j \\
&= \sum_{j=1}^{n} (M_j(f) - m_j(f))\Delta x_j \\
&= \sum_{j=1}^{n} (f(y_j) - f(z_j))\Delta x_j \\
&\leq \sum_{j=1}^{n} |f(y_j) - f(z_j)|\, \Delta x_j \\
&< \sum_{j=1}^{n} \frac{\epsilon}{b-a}\Delta x_j = \frac{\epsilon}{b-a}(b-a) = \epsilon
\end{aligned}$$

□

Ex 5.2. *The Dirichlet Function*

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

is not Riemann Integrable on $[0,1]$.

Proof. Take arbitrary paartition P of $[0,1]$, $M_j(f) = \sup_{[x_{j-1},x_j]} f = 1$ and $m_j = \inf_{[x_{j-1},x_j]} f = 0$, so $U(f,P) - L(f,P) = 1$. □

Definition 5.8 (Upper and Lower integral). $f : [a,b] \to \mathbb{R}$ *is bounded, then*

1. *the* ***upper integral*** *of* f *is*

$$(u)\int_a^b f(x)dx = \inf_P U(f,P)$$

2. *the* ***lower integral*** *of* f *is*

$$(l)\int_a^b f(x)dx = \sup_P L(f,P)$$

3. *define*

$$\int_a^b f(x)dx = (u)\int_a^b f(x)dx \quad \textit{if} \quad (u)\int_a^b f(x)dx = (l)\int_a^b f(x)dx$$

Remark 5.4.

$$(l)\int_a^b f(x)dx \leq (u)\int_a^b f(x)dx$$

Theorem 5.2. *Let* $f : [a,b] \to \mathbb{R}$ *be bounded, then* f *is integrable on* $[a,b]$ *iff*

$$(u)\int_a^b f(x)dx = (l)\int_a^b f(x)$$

5.2 Riemann Sum

Definition 5.9. *A **Riemann Sum** of f w.r.t. $P = \{x_0, \ldots, x_n\}$ of $[a,b]$ generated by samples $t_j \in [x_{j-1}, x_j]$ is*

$$S(f, P, t_j) = \sum_{j=1}^{n} f(t_j) \cdot \Delta x_j$$

*Which converges to $I(f)$ as $\|P\| \to 0$ if for **any** samples $t_j \in [x_{j-1}, x_j]$*

$$\forall \epsilon > 0.\ \exists P_\epsilon \subseteq [a,b].\ P = \{x_0, \ldots, x_n\} \supset P_\epsilon \Rightarrow |S(f, P, t_j) - I(F)| < \epsilon$$

Theorem 5.3. *$f : [a,b] \to \mathbb{R}$ bounded then f is integrable on $[a,b]$ iff $\lim_{\|P\| \to 0} S(f, P, t_j)$ exists.*

Theorem 5.4 (Linearity)**.** *If f, g are intergable on $[a,b]$, $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$ is also integrable, and moreover*

$$\int_a^b (\alpha f + \beta g)(x) dx = \alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx$$

Proof.

$$\begin{aligned}
\int_a^b f(x) dx &= \lim_{\|p\| \to 0} S(f, P, t_j) && (\forall t_j) \\
&= \lim_{\|p\| \to 0} \sum_{j=1}^{b} f(t_j) \Delta x_j && (\forall t_j) \\
\alpha \int_a^b f(x) dx &= \alpha \lim_{\|p\| \to 0} \sum_{j=1}^{b} f(t_j) \Delta x_j && (\forall t_j) \\
\beta \int_a^b g(x) dx &= \beta \lim_{\|p\| \to 0} \sum_{j=1}^{b} g(s_j) \Delta x_j && (\forall s_j) \\
\alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx &= \lim_{\|p\| \to 0} \sum_{j=1}^{b} \alpha f(t_j) \Delta x_j + \lim_{\|p\| \to 0} \sum_{j=1}^{b} \beta g(t_j) \Delta x_j && (\forall t_j) \\
&= \alpha \lim_{\|p\| \to 0} \sum_{i=1}^{n} (\alpha f + \beta g) \cdot t_j \cdot \Delta x_j && (\forall t_j) \\
&= S(\alpha f + \beta g, P, t_j) && (\forall t_j)
\end{aligned}$$

□

Theorem 5.5. *If f is integrable on $[a,b]$, then for all $c \in [a,b]$, f is integrable on $[a,c]$ and $[c,b]$, and moreover*

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

Theorem 5.6 (Comparison)**.** *If f, g are integrable on $[a,b]$ and $f \leq g$ on $[a,b]$, then*

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx$$

Corollary 5.2. *If $m \leq f(x) \leq M$ on $[a,b]$, then*

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$$

Theorem 5.7. *If f is integrable on $[a,b]$, so is $|f|$ and*

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|\,dx$$

Theorem 5.8. *If f,g are integrable on $[a,b]$, then so is fg.*

Theorem 5.9. *If f is integrable on $[a,b]$, $\forall [c,d] \subset [a,b]$. f is integrable on $[c,d]$.*

Theorem 5.10 (1st Mean Value Theorem)**.** *Let f,g be integrable functions on $[a,b]$. $g(x) \ge 0$ on $[a,b]$. If $m = \inf_{[a,b]} f$, $M = \sup_{[a,b]} f$, then*

$$\exists C \in [m,M].\ \int_a^b f(x)g(x)dx = C\int_a^b g(x)dx$$

In particular, if f is continuous on $[a,b]$, then

$$\exists x_0 \in [a,b].\ \int_a^b f(x)g(x)dx = f(x_0)\int_a^b g(x)dx$$

Proof. $g(x) \ge 0$ on $[a,b]$, $m \le f(x) \le M$ on $[a,b]$, then $mg(x) \le f(x)\cdot g(x) \le Mg(x)$ on $[a,b]$. By Comparison,

$$m\int_a^b g(x) = \int_a^b mg(x)dx \le \int_a^b f(x)g(x)dx \le \int_a^b Mg(x)dx = M\int_a^b g(x)dx$$

If $\int_a^b g(x)dx = 0$, then $\int_a^b g(x)dx = 0 = C\int_a^b g(x)dx, \forall C$. Otherwise,

$$m \le C = \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \le M$$

In particular, if f is continuous, by Intermidiate Value Theorem, $\exists x_0 \in [a,b]$. $f(x_0) = C$. □

Ex 5.3. $f(x) = \begin{cases} 1 & x \ge 0 \\ -1 & x < 0 \end{cases}$*, find $F(x) = \int_0^x f(t)dt$*

Answer. If $x \ge 0$, $F(x) = \int_0^x f(t)dt = x$; if $x < 0$, $F(x) = \int_0^x f(t)dt = -\int_x^0 f(t)dt = -x$. So $F(x) = |x|$.

Theorem 5.11. *If f is integrable on $[a,b]$, then $F(x) = \int_a^x f(t)dt$ exists and continuos on $[a,b]$.*

Proof. Since $[a,x] \subset [a,b]$, then $\int_a^x f(t)dt$ exists. Show $F(x)$ is continuous on $[a,b]$ by proving $\forall \epsilon > 0 \exists \delta > 0.\ \forall x_0 \in [a,b].\ |x - x_0| < \delta \Rightarrow |F(x) - F(x_0)| < \epsilon$ assume $\exists M > 0.\ |f| \le M$ on $[a,b]$. Choose $\delta = \frac{\epsilon}{M}$, assume $x_0 < x < x_0 + \delta$,

$$\begin{aligned}
|F(x) - F(x_0)| &= \left|\int_a^x f(t)dt - \int_a^{x_0} f(t)dt\right| \\
&= \left|\int_{x_0}^x f(t)dt\right| \\
&\le \int_{x_0}^x |f(t)|\,dt \\
&\le \int_{x_0}^x M dt \\
&= M(x - x_0) \\
&< M\delta = \epsilon
\end{aligned}$$

□

Theorem 5.12 (2nd Mean Value Theorem). *Let f, g be integrable in $[a, b]$. Assume $g \geq 0$ on $[a, b]$. Let $m \leq \inf_{[a,b]} f$, $M \geq \sup_{[a,b]} f$. Then*

$$\exists c \in [m, M]. \int_a^b f(x)g(x)dx = m \int_a^c g(x)dx + M \int_c^b g(x)dx$$

In particular, if $f \geq 0$ on $[a, b]$, then

$$\exists c. \in [a, b] \int_a^b f(x)g(x)dx = M \int_c^b g(x)dx$$

Proof. Use $m = 0$, then 2nd statement follows from the 1st statement. To prove 1st statement, define $h : [a, b] \to \mathbb{R}$ to be

$$h(x) = m \int_a^x g(x)dt + M \int_x^b g(t)dt$$

then h is continuous on $[a, b]$. $g \geq 0$, $m \leq f \leq M$, gives $m \cdot g \leq f \cdot g \leq M \cdot g$. By assumption,

$$h(b) = \int_a^b g(t)dt \leq \int_a^b f(t)g(t) \leq \int_a^b g(t)dt = h(a)$$

By IVT, there exists $c \in [a, b]$ s.t.

$$h(c) = m \int_a^x g(x)dt + M \int_x^b g(t)dt = \int_a^b f(x)g(x)dx$$

□

5.3 Fundamental Theorem of Calculus

Theorem 5.13 (FTC). $f : [a, b] \to \mathbb{R}$

1) *If f is continuous on $[a, b]$, define*

$$F(x) = \int_a^x f(t)dt$$

then $F \in C^1([a, b])$ and $\forall x \in [a, b]$. $F'(x) = f(x)$.

2) *If f is differentiable on $[a, b]$ and f' is integrable on $[a, b]$, then*

$$\forall x \in [a, b]. \int_a^x f'(t)dt = f(x) - f(a)$$

Proof of FTC

1) *Proof.* By symmetry, it suffices to show that if $f(x_0+) = f(x_0)$ then

$$\lim_{h \to 0^+} \frac{F(x_0 + h) - F(x_0)}{t} = f(x_0)$$

By definition, $\forall \epsilon > 0.\ \exists \delta > 0.\ x_0 < x < x_0 + \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$. Fix $0 < h < \delta$,

$$\begin{aligned} F(x_0 + h) - F(x_0) &= \int_a^{x_0+h} f(t)dt - \int_a^{x_0} f(t)dt \\ &= \int_{x_0}^{x_0+h} f(t)dt \end{aligned}$$

Note that

$$
\begin{aligned}
f(x_0) &= \frac{1}{h}\int_{x_0}^{x_0+h} f(x_0)dt \\
\implies \left|\frac{F(x_0+h)-F(x_0)}{h} - f(x_0)\right| &= \left|\frac{1}{h}\int_{x_0}^{x_0+h} f(t)dt - \frac{1}{h}\int_{x_0}^{x_0+h} f(x_0)dt\right| \\
&= \frac{1}{h}\left|\int_{x_0}^{x_0+h} (f(t)-f(x_0))dt\right| \\
&\leq \frac{1}{h}\int_{x_0}^{x_0+h} |f(t)-f(x_0)|\, dt \\
&\leq \frac{1}{h}\int_{x_0}^{x_0+h} \epsilon\ dt \\
&< \epsilon \\
\implies f(x_0) &= \lim_{h\to 0^+} \frac{F(x_0+h)-F(x_0)}{h}
\end{aligned}
$$

□

2) *Proof.* By definition,

$$\forall \epsilon > 0.\ \exists P = \{x_0, \dots, x_n\} \subseteq [a, x].\ \left|\sum_{j=1}^{n} f'(t_j)\cdot \Delta x_j - \int_a^x f'(t)dt\right| < \epsilon$$

By MVT (4.15 (ii) on P.111), which says if f is differentiable on $[a, b]$, then $\exists c \in [a, b].\ f(b) - f(a) = f'(c)(b - a)$, there exists $t_j \in [x_{j-1}, x_j]$ s.t.

$$
\begin{aligned}
f'(t_j)\cdot \Delta x_j &= f'(t_j)(x_j - x_{j-1}) \\
&= f(x_j) - f(x_{j-1}) \\
\implies \left|\sum_{j=1}^{n}(f(x_j) - f(x_{j-1})) - \int_a^x f'(t)dt\right| &< \epsilon
\end{aligned}
$$

Since

$$
\begin{aligned}
&\sum_{j=1}^{n}(f(x_j) - f(x_{j-1})) = f(x) - f(a) \\
&\implies \left|f(x) - f(a) - \int_a^x f'(t)dt\right| \\
&\iff f(x) - f(a) = \int_a^x f'(t)dt
\end{aligned}
$$

□

Remark 5.5. *The hypothesis in FTC cannot be relaxed.*

Theorem 5.14.

$$
\begin{aligned}
F(x) &= \int_a^{g(x)} g(t)dt \\
\frac{dF}{dx} &= f(g(x))\cdot g'(x)
\end{aligned}
$$

Proof. Define

$$G(u) = \int_a^u f(t)dt$$

then

$$\begin{aligned} F(x) &= G(g(x)) \\ \frac{dF}{dx} &= G'(g(x)) \cdot g'(x) \\ &\overset{\text{FTC}}{=} f(g(x)) \cdot g'(x) \end{aligned}$$

□

Theorem 5.15 (Intergration By Part)**.** *If:*

- f, g *are differentiable on* $[a, b]$
- f', g' *are integrable on* $[a, b]$

Then

$$\int_a^b f'(x)g(x)dx = f(x)g(x)|_a^b - \int_a^b f(x)g'(x)dx$$

Or

$$\int_a^b g \cdot df \quad = \quad (f \cdot g)\Big|_a^b - \int_a^b f \cdot dg$$

Proof.

$$\begin{aligned} (f(x) \cdot g(x))' &= f'(x) \cdot g(x) + f(x) \cdot g'(x) \\ \implies \quad \int_a^b (f(x) \cdot g(x))'dx &= \int_a^b f'(x) \cdot g(x) + f(x) \cdot g'(x)dx \\ f(x) \cdot g(x)|_a^b &= \int_a^b f'(x) \cdot g(x) + f(x) \cdot g'(x)dx \\ \iff \quad \int_a^b f'(x)g(x)dx &= f(x) \cdot g(x)|_a^b - f(x) \cdot g'(x)dx \end{aligned}$$

□

Theorem 5.16 (Change of Variables)**.** *Let* $\phi \in C^1([a, b])$ *and* F *be continuous on* $\phi([a, b])$*, then*

$$\int_{\phi(a)}^{\phi(b)} f(t)dt \xlongequal{t=\phi(x)} \int_a^b f(\phi(x))\phi'(x)dx$$

5.4 Improper Integral

Theorem 5.17. *Let* f *be integrable on* $[a, b]$*, then*

$$\int_a^b f(x)dx = \lim_{\substack{c \to a^+ \\ d \to b^-}} \int_c^d f(x)dx$$

Proof.

$$
\begin{aligned}
F(x) &= \int_a^x f(t)dt \text{ is continuous in } [a,b] \\
\int_a^b f(x)dx &= F(b) - F(a) \\
&= \lim_{d \to b^-} F(d) - \lim_{c \to a^+} F(c) \\
&= \lim_{\substack{c \to a^+ \\ d \to b^-}} \int_c^d f(x)dx
\end{aligned}
$$

□

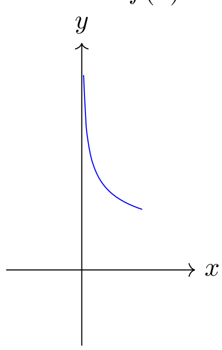
Definition 5.10 (Improper Integral)**.** *Let (a,b) be an open interval, possibly unbounded, $f:(a,b) \to \mathbb{R}$. Then*

- *f is **locally integrable** on (a,b) if f is integrable on each $[c,d] \subset (a,b)$*
- *f is **improperly integrable** if f is locally integrable on (a,b) and*

$$\int_a^b f(x)dx \overset{def}{=} \lim_{\substack{c \to a^+ \\ d \to b^-}} \int_c^d f(x)dx$$

*exists and is finite. The limit is called the **improper integral** of f on (a,b)*

Ex 5.4. *Show: $f(x) = x^{-\frac{1}{3}}$ is improperly integrable on $(0,1]$*



Proof. Since f is continuous on $(0,1]$, then f is integrable on every $[a,b] \in (0,1]$, or it's locally integrable.

$$
\begin{aligned}
\int_a^b f(x)dx &= \int_a^b x^{-\frac{1}{3}} dx \\
&= \frac{2}{3} x^{\frac{2}{3}} \Big|_a^b \\
&= \frac{3}{2}(b^{\frac{2}{3}} - a^{\frac{2}{3}}) \\
\lim_{a \to 0^+} \int_a^1 f(x)dx &= \lim_{a \to 0^+} \frac{3}{2}\left(1 - a^{\frac{2}{3}}\right) \\
&= \frac{3}{2}
\end{aligned}
$$

By definition,

$$\int_a^1 f(x)dx = \frac{3}{2}$$

□

Ex 5.5.

- *$f(x) = x^{-\alpha}$ is improperly integrable on $(0, 1)$ iff $\alpha < 1$*
- *$f(x) = x^{-\alpha}$ is improperly integrable on $(1, \infty)$ iff $\alpha > 1$*

Theorem 5.18 (Linearity)**.** *If f, g are improperly integrable on (a, b), then $\alpha f + \beta g$ is also improperly integrable on (a, b) and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$ is improperly integrable on (a, b) and*

$$\int_a^b (\alpha f + \beta g)(x)dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$$

Theorem 5.19 (Comparison)**.**

- *f, g: locally integrable on (a, b)*
- *$\forall x \in (a, b).\ 0 \leq f(x) \leq g(x)$*
- *g is improperly integrable on (a, b)*

then f is improperly integrable on (a, b) and

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

Ex 5.6. *Prove: $f(x) = \frac{\sin x}{\sqrt{x^3}}$ is improperly integrable on $(0, 1]$.*

Proof. $f \geq 0$ on $(0, 1]$. f is continuous gives that f is locally integrable on $(0, 1]$. Notice that $\sin x \leq x$ on $(0, 1]$, gives that

$$\frac{\sin x}{x^{\frac{3}{2}}} \leq \frac{x}{x^{\frac{3}{2}}} = x^{-\frac{1}{2}}$$

Since $x^{-\frac{1}{2}}$ is improperly integrable on $(0, 1]$, by comparison, f is improperly integrable on $(0, 1]$. □

Ex 5.7. *Prove $f(x) = \frac{\log x}{x^{\frac{5}{2}}}$ is improperly integrable on $[1, \infty)$.*

Proof. $f \geq 0$ on $[1, \infty)$. Since f is continuous on $[1, \infty)$, f is locally integrable on $[1, \infty)$. Notice that $\log x \leq x$ on $[1, \infty)$, gives that

$$\frac{\log x}{x^{\frac{5}{2}}} \leq \frac{x}{x^{\frac{5}{2}}} = x^{-\frac{3}{2}}$$

Since $x^{-\frac{3}{2}}$ is improperly integrable on $[1, \infty)$, by comparison, f is improperly integrable on $[1, \infty)$. □

Corollary 5.3. *Assume:*

- *f is bounded, locally integrable on (a, b)*
- *$|g|$ is locally integrable on (a, b)*

Then $|f \cdot g|$ is improperly integrable on (a, b)

Definition 5.11 (Absolute Integrability)**.** *f is **absolutely integrable** on (a, b) if f is locally integrable on (a, b) and $|f|$ is improperly integrable on (a, b).*

Definition 5.12 (Conditionally Integrability)**.** *f is **conditionally integrable** on (a, b) if f is locally integrable on (a, b) and $|f|$ is not improperly integrable on (a, b).*

Theorem 5.20. *If f is absolutely integrable then f is improperly integrable on (a, b) and*

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f|(x)dx$$

Ex 5.8. *$\frac{\sin x}{x}$ is conditionally integrable on $[0, \infty)$.*

Chapter 6

Infinite series of numbers

6.1 Introduction

Definition 6.1. *Let* $S = \sum_{k=1} a_k = a_1 + a_2 + a_3 + \dots,$

- $\forall n \in \mathbb{N}$, *the* ***partial sum of*** S ***of order*** n *is* $S_n = \sum_{k=1}^{n} a_k$.
- S *is* ***convergent*** *if* $\{S_n\}_{n=1}^{\infty}$ *converges. If* $S_n \to S$ *as* $n \to \infty$ *then* S *converges to* s*, denoted as* $\sum_{k=1}^{\infty} a_k = s$
- S *is* ***divergent*** *if* $\{S_n\}_{n=1}^{\infty}$ *is divergent. When* $\{S_n\}_{n=1}^{\infty}$ *diverges to* $\pm\infty$*, write* $\sum_{k=1}^{\infty} a_k = \pm\infty$.

Ex 6.1. $\sum_{k=1}^{\infty}(-1)^k$ *is divergent.*

Proof.
$$S_1 = -1, S_2 = 0, S_3 = -1, S_4 = 0, \dots \Rightarrow \{S_n\}_{n=1}^{\infty} = \{-1, 0, -1, 0, \dots\}$$
which is divergent. □

Ex 6.2. *Prove that* $\sum_{k=1}^{\infty} 2^{-k} = 1$

Proof.
$$S_n = \sum_{k=1}^{n} 2^{-k} = 1 - 2^{-n}$$
$$\lim_{n\to\infty} S_n = 1$$
□

Ex 6.3 (Harmonic Series)**.** *Prove:* $\sum_{k=1}^{\infty} \frac{1}{k}$ *diverges*

Proof.
$$\begin{aligned} S_n = \sum_{k=1}^{n} \frac{1}{k} > \sum_{k=1}^{n} \int_{k-1}^{k} \frac{1}{x+1} dx \\ = \int_0^n \frac{1}{x+1} dx \\ \lim_{n\to\infty} \int_0^n \frac{1}{x+1} dx = \int_0^\infty \frac{1}{x+1} dx \\ \Rightarrow \{S_n\} \text{ diverges} \\ \Rightarrow \sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges} \end{aligned}$$
□

Theorem 6.1 (Divergent Test). *Let $\{a_k\}$ be a sequence, with $a_k \not\to 0$ as $k \to \infty$. Then $\sum_{k=1}^{\infty} a_k$ diverges.*

Proof. Argue by contradiction. Suppose $\sum_{k=1}^{\infty} a_k$ converges. By definition, $\{S_n\}_{n=1}^{\infty}$ converges to $a \in \mathbb{R}$, gives that $a_k = S_k - S_{k-1} \to a - a = 0$ as $k \to \infty$. This is a contradiction. Then $\sum_{k=1}^{\infty} a_k$ diverges. □

Theorem 6.2 (Telescope Series). *Assume $a_k \to a$ as $k \to +\infty$, then*

$$\sum_{k=1}^{a}(a_k - a_{k+1}) = a_1 - a$$

Proof.

$$\begin{aligned} S_n &= \sum_{k=1}^{n}(a_k - a_{k+1}) \\ &= (a_1 - a_2) + (a_2 - a_3) + \cdots + (a_n - a_{n+1}) \\ &= a_1 - a_{n+1} \\ &\to a_1 - a \text{ as } n \to \infty \end{aligned}$$

By definition, $\sum_{k=1}^{\infty}(a_k - a_{k+1}) = a_1 - a$ □

Theorem 6.3 (Geometric Series). *$\sum_{k=n}^{\infty} x^k$ converges iff $|x| < 1$. In particular,*

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

Proof. When $|x| \geq 1$, then $x^k \not\to 0$ as $k \to \infty$. By definition, $\sum k = n^{\infty} x^k$ is divergent. When $|x| < 1$, then

$$\begin{aligned} (1-x)S_n &= (1-x)(1 + x + x^2 + \cdots + x^n) \\ &= 1 + x + x^2 + \cdots + x^n - (x + x^2 + \cdots + x^{n+1}) \\ &= 1 - x^{n+1} \to 1 \text{ as } n \to \infty \\ &\Rightarrow \sum_{k=n}^{\infty} x^k \text{ converges and } \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \end{aligned}$$

□

Theorem 6.4 (Cauchy Criterion). *$\sum_{k=1}^{\infty}$ converges iff*

$$\forall \epsilon > 0.\ \exists N \in \mathbb{N}.\ \left|\sum_{k=m}^{n} a_k\right| < \epsilon, \quad \forall n \geq m \geq N$$

Theorem 6.5. *Let $\sum_{k=1}^{\infty} a_k, \sum_{k=1}^{\infty} b_k$ be convergent series. Then: $\sum_{k=1}^{\infty}(\alpha a_k + \beta b_k)$ converges and*

$$\sum_{k=1}^{\infty}(\alpha a_k + \beta b_k) = \sum_{k=1}^{\infty} \alpha a_k + \sum_{k=1}^{\infty} \beta b_k$$

6.2 Series with non-negative terms

Theorem 6.6. *Assume $a_k \geq 0$, then the series $\sum_{k=0} a_k$ converges iff $\{S_n\}$ is bounded, namely*

$$\forall n.\ \exists M > 0.\ \left|\sum_{k=1}^{n} a_k\right| \leq M$$

Proof.

- "$\Rightarrow$": If $\sum_{k=1} a_k$ converges, by definition, $\{S_n\}$ converges.
- "$\Leftarrow$": Note that $\{S_n\}$ is increasing. If $\{S_n\}$ is bounded, by monotone convergence theorem, $\{S_n\}$ converges. By definition, $\sum_{k=1} a_k$ converges.

□

Theorem 6.7 (Integral Test). *Let $f : [1,\infty) \to \mathbb{R}$ positive, decreasing, then $\sum_{k=1}^{\infty} f(k)$ converges iff f is improperly integrable on $[0,\infty)$, i.e.,*

$$\int_1^\infty f(x)dx < \infty$$

Proof. Let

$$\begin{aligned}
S_n &= \sum_{k=1}^{n} f(k) \\
t_n &= \int_1^n f(x)\,dx \\
f(k-1) \geq f(k) &\geq f(k+1) && (f \text{ decreasing}) \\
f(k) &\geq \int_k^{k+1} f(x)\,dx \geq f(k+1) && \text{(Comparison)}
\end{aligned}$$

Sum $k = 1,2,3,\ldots,n-1$

$$\begin{aligned}
\underbrace{\sum_{2}^{n} f(k)}_{S_n - f(1)} \leq \underbrace{\int_1^n f(x)\,dx}_{t_n} &\leq \underbrace{\sum_{k=1}^{n-1} f(k)}_{S_n - f(n)} \\
0 \leq f(n) \leq S_n - t_n &\leq f(1)
\end{aligned}$$

If $\sum_{k=1} f(k)$ converges, then $\sum_{k=1} f(k)$ is bounded

$$\begin{aligned}
t_n &= \int_1^n f(x)\,dx \text{ is bounded} \\
\int_1^\infty f(x)\,dx &= \text{converge by MCT}
\end{aligned}$$

If $\int_1^\infty f(k)\,dx < \infty$, then $t_n = \int_1^\infty g(x)dx < \infty$

$$\begin{aligned}
&S_n \text{ is bounded} \\
&\sum_{k=1}^{\infty} f(k) \text{ converges}
\end{aligned}$$

□

Corollary 6.1 (p-series).

$$\sum_{k=1}^{\infty} k^{-p} \textit{ converges iff } p > 1$$

Proof. Let $f(x) = k^{-p}$. When $p \leq 0$, $k^{-p} \not\to 0$, as $k \to \infty$. By diverge test,

$$\sum_{k=1}^{\infty} k^{-p} \text{ diverges}$$

When $p > 0$, $f(x)$ is positive and decreasing. Note

$$\int_1^\infty f(x)dx = \int_1^\infty \frac{1}{x^p}dx$$

converges iff $p > 1$. By integral test, $\sum_{k=1}^\infty k^{-p}$ converges iff $p > 1$. ☐

Theorem 6.8 (Comparison Test). *Given $0 \le a_k \le b_k$.*

- *If $\sum_k b_k$ converges, then $\sum_k a_k$ converges.*
- *If $\sum_k a_k$ diverges, then $\sum_k b_k$ diverges.*

Ex 6.4. *Determine*

$$\sum_{k=1}^\infty \frac{3k}{k^2+k}\sqrt{\frac{\log k}{k}}$$

converges or not

Proof. Notice that $\forall k \ge 1.\ \log k \le 2k^{\frac{1}{2}}$ because $\log 1 \le 2 \cdot 1^{\frac{1}{2}}$ and $(\log k)' = \frac{1}{k} < (2 \cdot k^{\frac{1}{2}})' = k^{-\frac{1}{2}}$. This gives that

$$\begin{aligned} \frac{3k}{k^2+k}\sqrt{\frac{\log k}{k}} &\le \frac{3k}{k^2}\sqrt{\frac{2k^{\frac{1}{2}}}{k}} \\ &= 3\sqrt{2} \cdot \frac{1}{k^{\frac{5}{4}}} && \text{(Converges by p-series)} \end{aligned}$$

By comparison, it converges. ☐

Theorem 6.9 (Limit Comparison Test). *Let $a_k \ge 0, b_k \ge 0$. Assume $L = \lim_{k\to\infty} \frac{a_k}{b_k} (\ge 0)$.*

1. *If $0 < L < \infty$, then $\sum_{k=1}^\infty a_k$ converges iff $\sum_{k=1}^\infty b_k$ converges.*
2. *If $L = 0$, and $\sum_{k=1}^\infty b_k$ converges, then $\sum_{k=1}^\infty a_k$ converges.*
3. *If $L = \infty$, and $\sum_{k=1}^\infty b_k$ diverges, then $\sum_{k=1}^\infty a_k$ diverges.*

Proof.

1. Fix $\epsilon = \frac{L}{2}$, by definition of limit,

$$\begin{aligned} \exists N \in \mathbb{N}.\ \left|\frac{a_k}{b_k} - L\right| &< \frac{L}{2} \\ \frac{L}{2} \le \frac{a_k}{b_k} &\le \frac{3L}{2} && (k \ge N) \\ \frac{L}{2} \cdot b_k \le a_k &\le \frac{3L}{2} \cdot b_k && (k \ge N) \\ \sum_{k=1}^\infty a_k &\text{ converges} && \text{(Comparison)} \end{aligned}$$

2. If $\frac{a_k}{b_k} \to 0$ as $k \to \infty$, then

$$\begin{aligned} \exists N.\ \frac{a_k}{b_k} &< 1 && (k \ge N) \\ a_k &< b_k && (k \ge N) \\ \sum_{k=1}^\infty b_k \text{ converges} &\Rightarrow \sum_{k=1}^\infty a_k \text{ converges} && \text{(Comparison)} \end{aligned}$$

3. If $\frac{a_k}{b_k} \to \infty$ as $k \to \infty$, then $\exists N.\ a_k > b_k$ as $k > N$. By Comparion, $\sum_{k=1}^{\infty} b_k$ converges gives that $\sum_{k=1}^{\infty} a_k$ converges.

□

Ex 6.5. *Determine converge or not:*

$$\sum_{k=1}^{\infty} \frac{k}{\sqrt{2k^4 + k^2 - k}}$$

Answer. *Let* $a_k = \frac{1}{k}$

$$\frac{\frac{k}{\sqrt{2k^4+k^2-k}}}{\frac{1}{k}} = \frac{k^2}{\sqrt{2k^4 + k^2 - k}} \to \frac{1}{\sqrt{2}} \text{ as } k \to \infty$$

Since $\sum_{k=1}^{\infty} \frac{1}{k}$ *diverges, by limit comparison test*

$$\sum_{k=1}^{\infty} \frac{k}{\sqrt{2k^4 + k^2 - k}} \qquad \text{diverges}$$

Ex 6.6. *Let* $a_k \to 0$ *as* $k \to \infty$, *Prove that* $\sum_{k=1}^{\infty} \sin|a_k|$ *converges iff* $\sum_{k=1}^{\infty} |a_k|$ *converges.*

Proof. Recall $\frac{\sin x}{x} \to 1$ as $x \to 0$. Because $|a_k| \geq 0$, $\sin|a_k| \geq 0$ as $k \to \infty$. Since $\frac{\sin|a_k|}{|a_k|} \to 1$ as $k \to \infty$ by limit comparison test.

$$\sum_{k=1}^{\infty} \sin|a_k| \text{ converges iff } \sum_{k=1}^{\infty} |a_k| \text{ converges}$$

□

6.3 Absolute Convergence

Definition 6.2. $S = \sum_{k=1}^{\infty} a_k$,

1. S ***converges absolutely*** *if*

$$\sum_{k=1}^{\infty} |a_k| < \infty$$

2. S ***converges conditionally*** *if* S *converges but not converge absolutely.*

Ex 6.7.

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$

converge conditionally.

Theorem 6.10. *If* $\sum_{k=1}^{\infty} a_k$ *converges absolutely, then* $\sum_{k=1}^{\infty} a_k$ *converges.*

Definition 6.3 (Limit Supremum)**.** *Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence,*

$$\limsup_{k\to\infty} x_k = \lim_{n\to\infty} \left\{ \sup_{k>n} x_k \right\} \in \bar{\mathbb{R}}$$

Ex 6.8. $X_k = k$,

$$\limsup_{k\to\infty} k = \lim \left\{ \sup_{k>n} k \right\} = \lim_{n\to\infty} \infty = \infty$$

Ex 6.9.

$$\begin{aligned}\limsup_{k\to\infty}(-k) &= \lim_{n\to\infty}\left(\sup_{k>n}(-k)\right)\\ &= \lim_{n\to\infty} -(n+1)\\ &= -\infty\end{aligned}$$

Ex 6.10.

$$\begin{aligned}\limsup_{k\to\infty}\frac{1+(-1)^k}{k} &= \lim_{k\to\infty}\left\{0,1,0,\frac{1}{2},0,\frac{1}{3},0,\frac{1}{4},\dots\right\}\\ &= \lim_{n\to\infty}\left\{\sup_{k>n}\frac{1+(-1)^k}{k}\right\}\\ &= 0\end{aligned}$$

Fact 6.1. $\limsup_{k\to+\infty} a_k$ *is the largest possible limit among all convergent subsequences of* $\{x_k\}_{k=1}^{\infty}$.

Proposition 6.1.

1. *If* $\limsup_{k\to+\infty} x_k < x$, *then* $x_k < x$ *for large* k.
2. *If* $\limsup_{k\to+\infty} x_k > x$, *then* $x_k > x$ *for infinitely many* k.
3. *If* $x_k \to x$ *as* $k \to +\infty$, *then* $\limsup_{k\to\infty} x_k = x$.

Theorem 6.11 (Root Test). *Let* $r = \limsup_{k\to\infty} |a_k|^{\frac{1}{k}}$.

1. *If* $r < 1$, *then* $\sum_{k=1}^{\infty} a_k$ *converges absolutely*
2. *If* $r > 1$, *then* $\sum_{k=1}^{\infty} a_k$ *diverges*

Theorem 6.12 (Ratio Test). *Let* $r = \lim_{k\to\infty}\frac{|a_k+1|}{|a_k|}$,

1. *If* $r < 1$, *then* $\sum_{k=1}^{\infty} a_k$ *converges absolutely*
2. *If* $r > 1$, *then* $\sum_{k=1}^{\infty} a_k$ *diverges*

6.4 Alternating Series

Lemma 6.1 (Abel's Lemma). *Let* $\{a_k\}_{k\in\mathbb{N}}, \{b_k\}_{k\in\mathbb{N}}$ *be real sequences.* $\forall n \geq m > 1$, *set* $A_{n,m} = \sum_{k=m}^{n} a_k$. *Then*

$$\sum_{k=m}^{n} a_k b_k = A_{n,m}b_n - \sum_{k=m}^{n-1} A_{k,m}(b_{k+1}-b_k)$$

Theorem 6.13 (Dirichlet Test). $a_k, b_k \in \mathbb{R}$. *If* $S_n = \sum_{k=1}^{n} a_k$ *is bounded and* $b_k \downarrow 0$ *as* $k \to 0$. *Then*

$$\sum_{k=1}^{\infty} a_k b_k$$

converges

Proof. Choose $M > 0$ s.t.

$$\forall n \in \mathbb{N}.\ |S_n| = \left|\sum_{k=1}^{n} a_k\right| \leq \frac{M}{2}$$

$$|A_{n,m}| = |S_n - S_{m-1}| \leq |S_n|$$

$$\forall n \geq m \geq 1.\ |S_{m-1}| \leq \frac{M}{2} + \leq \frac{M}{2} = M$$

$\forall \epsilon > 0. \exists N \in \mathbb{N}. 0 \le b_k < \frac{\epsilon}{M}$ as $k \ge N$

$$\begin{aligned}
\left|\sum_{k=m}^{n} a_k b_k\right| &= \left|A_{n,m} b_n - \sum_{k=m}^{n-1} A_{k,m}(b_{k+1} - b_k)\right| \\
&\le |A_{n,m} b_n| + \sum_{k=m}^{n-1} |A_{k,m}|\,|b_{k+1} - b_k| \\
&= |A_{n,m} b_n| + \sum_{k=m}^{n-1} |A_{k,m}|\,(b_k - b_{k+1}) \\
&\le M \cdot \frac{\epsilon}{M} + \sum_{k=m}^{n-1} M \cdot (b_k - b_{k+1}) && (n \ge m > N) \\
&= \epsilon + M(b_m - b_n) \\
&\le \epsilon + M \cdot b_m \le \epsilon + M \cdot \frac{\epsilon}{M} = 2\epsilon
\end{aligned}$$

By Cachy Criterion

$$\sum_{k=1}^{\infty} a_k b_k \text{ converges}$$

□

Corollary 6.2 (Alternating Series Test). *If $b_k \downarrow 0$ as $k \to \infty$, then*

$$\sum_{k=1}^{\infty} (-1)^k b_k$$

converges

Ex 6.11. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ *converges conditionally*

Ex 6.12. $\sum_{k=1}^{\infty} \frac{(-1)^k}{\log k}$ *converges conditionally*

General Procedure. Let $S = \sum_{k=1}^{\infty} a_k$,

Step 1. Apply divergent test. Namely, verify if $a_k \not\to 0$ as $k \to \infty$

Step 2. Determine if S is geometric or p-series or if S looks like geometric or p-series; apply comparison test or limit comparison test to $\sum_{k=1}^{\infty} |a_k|$

Step 3. Apply root test or ratio test. Namely, find $r = \limsup_{k\to+\infty} |a_k|^{\frac{1}{k}}$ or $r = \lim_{k\to\infty} \frac{|a_{k+1}|}{a_k}$. If $r < 1$ then S converges absolutely. If $r > 1$, S diverges.

Step 4. If S alternates, apply alternating series test.

Chapter 7

Infinite Series of Functions

7.1 Uniform Converges of Sequences

Definition 7.1. $\{f_n\}$ ***converges uniformly*** *to f on E if*

$$\forall \epsilon > 0.\ \exists N \in \mathbb{N}.\ \forall n \geq N.\ |f_n - f| < \epsilon$$

Theorem 7.1. *Assume $f_n \to f$ uniformly on E and assume each f_n is continuous on E, then f is continuous on E*

Theorem 7.2. *Assume $f_n \to f$ uniformly on $[a, b]$ and assume each f_n is integrable on $[a, b]$, then f is integrable on $[a, b]$. Moreover,*

$$\int_a^b f(x)\ dx = \lim_{n \to +\infty} \int_a^b f_n(x)\ dx$$

Lemma 7.1 (Uniform Cauchy Criterion)**.** *$f_n \to f$ uniformly on E as $n \to +\infty$ iff*

$$\forall \epsilon > 0.\ \exists N \in \mathbb{N}.\ \forall i, j \geq N.\ |f_i - f_j| < \epsilon$$

Theorem 7.3. *Assume*

1. *(a, b) is bounded open interval*
2. *f_n is differentiable on (a, b) for every n*
3. *$\exists x_0 \in (a, b)$. such that $f_n(x_0)$ converges*
4. *f_n' converges uniformly on (a, b)*

then $f_n \to f$ to f on (a, b), and

$$\forall x \in (a, b).\ f'(x) = \lim_{n \to \infty} f_n'(x)$$

Proof. Fix $c \in (a, b)$, define

$$g_n(x) = \begin{cases} \frac{f_n(x) - f_n(c)}{x - c} & x \neq c \\ f_n'(c) & x = c \end{cases}$$

then g_n is continuous on (a, b).

Lemma 7.2. *$g_n(x)$ converges uniformly on (a, b)*

Proof.

1. When $x \neq c$,

$$\begin{aligned} g_n(x) - g_m(x) &= \frac{f_n(x) - f_n(c)}{x - c} - \frac{f_m(x) - f_m(c)}{x - c} \\ &= \frac{(f_n - f_m)(x) - (f_n - f_m)(c)}{x - c} \\ &= (f_n - f_m)'(\xi), \quad \xi \in (x, c) \\ \Rightarrow |g_n(x) - g_m(x)| &= |f_n'(x) - f_m'(x)| < \epsilon, \quad n, m \geq N \end{aligned}$$

2. When $x = c$

$$|g_n(c) - g_m(c)| = |f_n'(c) - f_m'(c)| < \epsilon, \quad n, m \geq N$$

Combining 1 and 2,

$$\forall x \in (a, b).\ \forall n, m \geq N.\ |g_n(x) - g_m(x) < \epsilon|$$

then g_n converges uniformly on (a, b). □

Let $x_0 = c$,

$$f_n(x) = g_n(x)(x - x_0) + f_n(x_0)$$

then

$$\begin{aligned} |f_n(x) - f_m(x)| &= |[g_n(x)(x - x_0) + f_n(x_0)] - [g_m(x)(x - x_0) + f_m(x_0)]| \\ &\leq |x - x_0| \cdot |g_n(x) - g_m(x)| + |f_n(x_0) - f_m(x_0)| \\ &< (b - a) \cdot \epsilon + \epsilon \\ &= (b - a + 1) \cdot \epsilon \end{aligned}$$

By uniform Cauchy criterion, f_n converges uniformly on (a, b). □

7.2 Uniforma Converges of Series

Definition 7.2 (Series of Functions)**.**

$$S(x) = \sum_{k=1}^{\infty} f_k(x)$$

Definition 7.3 (Pointwise Convergence)**.** *The series $\sum_{k=1}^{\infty} f_k(x)$ **converges pointwisely** on E iff*

$$\forall x \in E.\ \sum_{k=1}^{\infty} f_k(x) \text{ converges}$$

or $S_n(x)$ converges as $n \to \infty$

Definition 7.4 (Uniform Convergence)**.** *The series $\sum_{k=1}^{\infty} f_k(x)$ **converges uniformly** on E iff*

$$\forall x \in E.\ \sum_{k=1}^{\infty} f_k(x) \text{ converges uniformly}$$

Theorem 7.4 (Weierstrass M-test). *$f_k : E \to \mathbb{R}$. Suppose $\exists M_k > 0.\ |f_k| \leq M_k$ on E,*

$$\sum_{k=1}^{\infty} M_k \text{ converges} \Longrightarrow \sum_{k=1}^{\infty} f_k(x) \text{ converges uniformly on } E$$

Proof. By definition, $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly on E if $S_n(x)$ converges uniformly on E. Prove by using uniform Cauchy criterion:

$$\forall \epsilon > 0.\ \exists N \in \mathbb{N}.\ \forall m > n \geq N.\ |S_m(x) - S_n(x)| < \epsilon \iff \left| \sum_{k=n+1}^{m} f_k(x) \right| < \epsilon$$

Since $\sum_{k=1}^{\infty} M_k < \infty$, by the Cauchy criterion,

$$\forall \epsilon > 0.\ \exists N \in \mathbb{N}.\ \forall m > n \geq N.\ \left| \sum_{k=n+1}^{m} M_k \right| < \epsilon$$

Because $|f_k| \leq M_k$, then

$$\left| \sum_{k=n+1}^{m} f_k(x) \right| \leq \sum_{k=n+1}^{m} |f_k(x)| \leq \sum_{k=n+1}^{m} M_k < \epsilon$$

□

Definition 7.5 (Absolute Convergence). *The series $\sum_{k=1}^{\infty} f_k(x)$ **converges absolutely** on E iff*

$$\forall x \in E.\ \sum_{k=1}^{\infty} |f_k(x)| \text{ converges}$$

Theorem 7.5. *$f_k : E \to \mathbb{R}$. If f_k is continuous at $x_0 \in E$ and $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly on E, then f is continuos at x_0.*

Theorem 7.6 (Turn by turn integration). *$f_k : [a, b] \to \mathbb{R}$. If f_k is integrable on $[a, b]$ and $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly on $[a, b]$, then*

$$\int_a^b f(x)\, dx = \sum_{k=1}^{\infty} \int_a^b f_k(x)\, dx$$

Theorem 7.7 (Turn by turn differentiation). *$f_k : (a, b) \to \mathbb{R}$. If f_k is differentiable on (a, b) and $\sum_{k=1}^{\infty} f_k(x)$ converges at $x_0 \in (a, b)$, and $\sum_{k=1}^{\infty} f_k'(x)$ converges uniformly to f on (a, b), f is differentiable and*

$$\forall x \in (a, b).\ f'(x) = \sum_{k=1}^{\infty} f_k'(x)$$

Ex 7.1. *Show $\sum_{k=1}^{\infty} \frac{x^k}{k}$ converges uniformly on $\left[-\frac{1}{3}, \frac{1}{2}\right)$*

Proof. Notice that

$$\forall k \in \mathbb{N}.\ \left| \frac{x^k}{k} \right| \leq \left| x^k \right| \leq \left| \frac{1}{2} \right|^k$$

and

$$\sum_{k=1}^{\infty} \left| \frac{1}{2} \right|^k \text{ converges}$$

By Weierstrass M-test, $\sum_{k=1}^{\infty} \frac{x^k}{k}$ converges uniformly on $\left[-\frac{1}{3}, \frac{1}{2}\right)$ □

7.3 Power Series

Definition 7.6 (Power Series). *Given $a_k, x_0 \in \mathbb{R}$*

$$S(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

Definition 7.7. *A non-negative extended real number R is called the* ***radius of convergence*** *of a power series $S(x)$ if $S(x)$ converges absolutely for $|x - x_0| < R$ and diverges for $|x - x_0| > R$.*

Ex 7.2. *Consider special case*

$$\sum_{k=0}^{\infty} k^k x^k$$

in which $a_k = k^k$, $x_0 = 0$. By root test,

$$\limsup_{k \to +\infty} \left| k^k x^k \right|^{\frac{1}{k}} = \limsup_{k \to \infty} k \, |x|$$

For it to converge, it requires $kx < 1$ as $k \to \infty$, namely $x = 0$. Therefore, the series converges only at $x = 0$.

Theorem 7.8 (Root Test). *Given power series $S(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$. Let*

$$R = \frac{1}{\limsup_{k \to \infty} |a_k|^{\frac{1}{k}}}$$

then R is the radius of convergence for $S(x)$. Moreover,

1. *$S(x)$ converges absolutely $\forall x \in (x_0 - R, x_0 + R)$*
2. *$S(x)$ converges uniformly on $[a, b] \subset (x_0 - R, x_0 + R)$*
3. *When $R < +\infty$, $S(x)$ diverges for $x \notin [x_0 - R, x_0 + R]$*

Proof. Define

$$R = \frac{1}{\limsup_{k \to \infty} |a_k|^{\frac{1}{k}}}$$

Apply root test to $S(x)$, define

$$\begin{aligned} r(x) &= \limsup_{k \to \infty} \left| a_k (x - x_0)^k \right|^{\frac{1}{k}} \\ &= |x - x_0| \cdot \limsup_{k \to \infty} |a_k|^{\frac{1}{k}} \\ &= \frac{|x - x_0|}{R} \end{aligned}$$

Case 1. $R = 0$, then

$$r(x) = \begin{cases} +\infty & \text{if } x \neq x_0 \\ 0 & \text{if } x = x_0 \end{cases}$$

Then the theorem is proved

Case 2. $R = +\infty$, then $r(x) = 0$. By root test, $S(x)$ converges absolutely for all $x \in \mathbb{R}$.

Case 3. $0 < R < \infty$. Then by root test, $S(x)$ converges absolutely if $r(x) < 1 \iff |x - x_0| < R$ and diverges if $r(x) > 1 \iff |x - x_0| > R$

Suppose $[a, b] \subset (x_0 - R, x_0 + R)$, then

$$\exists 0 < c < 1.\ \forall x \in [a, b].\ a_k \left|(x - x_0)^k\right| \leq c^k$$

By Weierstrass M-test, $S(x)$ converges uniformly on $[a, b]$. □

Theorem 7.9 (Ratio Test). *If $\rho = \lim_{k\to\infty} \frac{|a_k|}{|a_{k+1}|}$ exists as an extended real number, the ρ is the radius of convergence for a power series.*

Definition 7.8 (Interval of converges). *The **interval of convergence** of a a power series $S(x)$ is the interval where $S(x)$ converges.*

Remark 7.1.

1. *If $R = 0$, then $I = \{x_0\}$*
2. *If $R = +\infty$, then $I = \mathbb{R}$*
3. *If $0 < R < +\infty$, then I is $(x_0 - R, x_0 + R)$, $[x_0 - R, x_0 + R]$, $(x_0 - R, x_0 + R]$ or $[x_0 - R, x_0 + R)$. Consider $x = x_0 - R$ and $x = x_0 + R$.*

Ex 7.3. *Find the interval of convergence of*

$$S(x) = \sum_{k=1}^{\infty} \frac{x^k}{\sqrt{k}}$$

By ratio test,

$$R = \lim_{k\to\infty} \frac{\frac{1}{\sqrt{k}}}{\frac{1}{\sqrt{k+1}}} = 1$$

When $x = -1$,

$$S(-1) = \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$$

converge by alternating series test. When $x = 1$,

$$S(1) = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$$

diverges by p-series test. Therefore, the interval of convergence is $[-1, 1)$.

Ex 7.4. *Find the inteval of convergence of*

1. *$\sum_{k=1}^{\infty} x^k$. Apply ratio test: $R = 1$. When $x = -1$, $\sum_{k=1}^{\infty} (-1)^k$ diverges by alternating series test. When $x = 1$, $\sum_{k=1}^{\infty} 1$ diverges by p-series test. Therefore, the interval of convergence is $(-1, 1)$*
2. *$\sum_{k=1}^{\infty} \frac{x^k \cdot (-1)^k}{k}$. Apply ratio test:*

$$\lim_{k\to\infty} \frac{\left|\frac{(-1)^{k+1}}{k+1}\right|}{\left|\frac{(-1)^k}{k}\right|} = \lim_{k\to\infty} \frac{k}{k+1} = 1$$

   When $x = -1$, $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges by p-series test. When $x = 1$, $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges by alternating series test. Therefore, the interval of convergence is $(1, -1]$
3. *$\sum_{k=1}^{\infty} \frac{x^k}{k^2}$. $[-1, 1]$*