Introduction to Topology

MAT 661

George Miao
gm@miao.dev

Contents

Chapter 1

Basic Point Set Topology

1.1 Real

Definition (Open balls). $x \in \mathbb{R}^n, r > 0$, $B(x, r) := \{y \in \mathbb{R}^n \mid d(x, y) < r\}$

Definition (Open set). $u \subseteq \mathbb{R}^n$ is open if every $x \in u$ an **interior point** of u, meaning $\exists r > 0.\ B(x, r) \subseteq u$.

Remark. r-balls are open.

Theorem (Continuity). Let $f : \mathbb{R}^k \to \mathbb{R}^l$. f is continuous iff for every open set $v \subseteq R^l$, the preimage $f^{-1}(v)$ is open in $\mathbb{R}^l$.

proof.
- "⇒" skipped
- "⇐" Suppose preimages of opensets are open, and let $x \in \mathbb{R}^k$ and $\varepsilon > 0$. Then $B(f(x), \varepsilon)$ is open in $\mathbb{R}^l$, so by assumption,

$$f^{-1}(B(f(x), \varepsilon)) \text{ is open in } \mathbb{R}^k$$

⇒ x is an interior point of $f^{-1}(B(f(x), \varepsilon))$
⇒ $\exists \delta > 0.\ B(x, \delta) \subseteq f^{-1}(B(f(x), \varepsilon))$
⇒ ε-δ condition holds at x
⇒ f is continuous at x
⇒ f is continuous on all of $\mathbb{R}^k$

□

Definition. $X \subseteq \mathbb{R}^k$, $Y \subseteq \mathbb{R}^l$ subsets, $f : X \to Y$. f is continuous at $x \in X$ if

$$\forall \varepsilon > 0.\ \exists \delta > 0.\ f(B_X(x, \delta)) \subseteq B_Y(f(x), \varepsilon)$$

where $B_X(x, \delta) := B(x, \delta) \cap X$ and $B_Y(y, \varepsilon) := B(y, \varepsilon) \cap Y$.

Definition . $X \subseteq \mathbb{R}^n$, $U \subseteq X$ subset. U is **open in $\boldsymbol{X}$** if there exists an open set $U' \subseteq \mathbb{R}^n$ such that

$$U = U' \cap X$$

Example. Let

$$X = [0,2] \times [0,2]$$
$$U = \{(x_1, x_2) \in X \mid x_1^2 + x_2^2 < 1\}$$

U is open in $\mathbb{R}^2$ because $U = B((0,0),2) \cap X$.

Example. For every $X \subseteq \mathbb{R}^n$, X is open in X because $X = X \cap \mathbb{R}^n$. But in geenral, $X \subseteq \mathbb{R}^n$ is **not** open in $\mathbb{R}^n$.

Theorem. Let $X \subseteq \mathbb{R}^k$, $Y \subseteq \mathbb{R}^l$, $f : X \to Y$. f is continuous iff for every $V \subseteq Y$ that's open in Y, the preimage $f^{-1}(V)$ is open in X.

1.2 Metric Spaces

Definition. X any set. A **distance function** or **metric** on X is a map

$$d : X \times X \to [0, \inf)$$

such that
(M1) $d(x,y) = 0 \iff x = y$
(M2) $d(x,y) = d(y,x)$
(M3) $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality)

Definition (Metric space). (X, d) is a metric space if d is a metric on X.

Example. $(\mathbb{R}^n, d)$ is a metric where d is the Euclidean distance, i.e. $d(x,y) = \|x - y\|$.

Example. $(\mathbb{R}^n, d')$ where $d' = 2d$ is also a metric space.

Example (Discrete metric). X any set, $d = d_{\text{discret}}$ where

$$d_{\text{discrete}(x,y)} = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

(X, d_{discrete}) is called the **discrete metric space**.

Example. (X, d) any metric space, $Y \subseteq X$ subset, we can restrict d to a map

$$d|_Y : Y \times Y \to [0, \inf)$$

then $(Y, d|_Y)$ is a metric space, called a **(metric) subspace** of (x, d) and $d|_Y$ called **induced metric**.

Example. S is a subspace in $(\mathbb{R}^3, d_{\text{Eucl}})$, then $(S, d_{\text{Eucl}}|_S)$ is a metric space.

Example. Let V be a real vector space. A norm on V is a map $\|.\| : V \to [0, \inf)$ such that
(N1) $\|x\| = 0 \iff x = 0$
(N2) $\|cx\| = |c| \; \|x\|$
(N3) $\|x + y\| \leq \|x\| + \|y\|$

e.g., on $V = \mathbb{R}^2$,

- Euclidean norm: $\|(x_1, x_2)\| = \sqrt{x_1^2 + x_2^2}$
- Max norm: $\|(x_1, x_2)\| = \max(|x_1|, |x_2|)$
- Sum norm: $\|(x_1, x_2)\| = |x_1| + |x_2|$

Easy to see: If $\|.\|$ is a norm on V, then $d(x, y) = \|x - y\|$ is a metric on V, meaning any normed vector space is a metric space.

Definition (Open balls)**.** Let (X, d) be metric space, $x \in X$, $r > 0$. The **open d-r-ball centered at x** is the set

$$B_d(x, r) := \{y \in X \mid d(x, y) < r\}$$

Definition (Open set)**.** Let (X, d) be metric space. A subset $U \subseteq X$ is **open** if every $x \in U$ is an **interior point** of U, meaning

$$\exists r > 0. \; B_d(x, r) \subseteq U$$

Definition (Continuity)**.** Let (X, d), (Y, d') be metric spaces, $f : X \to Y$. f is continuous at $x \in X$ if

$$\forall \varepsilon > 0. \; \exists \delta > 0. \; f(B_d(x, \delta)) \subseteq B_{d'}(f(x), \varepsilon)$$

Theorem. Let (X, d), (Y, d') be metric spaces, $f : X \to Y$. f is continuous iff the preimage of d'-open set $V \subseteq Y$ is d-open in X.

Theorem. Let (X, d) be metic space
1. $\emptyset, X$ are open (in X)
2. the union of any collection of open sets in X is open
3. the intersection of any **finite** collection of open sets in X is open

proof.

1. $\emptyset$ is open because it contains no non-interior points. X is open because **every** $B_d(x, r)$ is contained in X.
2. Suppose the sets $U_i, i \in I$ are open in X, and $x \in \bigcup U_i$, then $\exists i \in I. \; x \in U_i$, meaning x is an interior point of U_i for some i. So $\exists r > 0. \; B_d(x, r) \subseteq U_i \subseteq \bigcup U_i$.

3. Suppose $U_1, ..., U_n$ are open subsets of X. Let

$$x \in \bigcap_{i \in [1,n]} U_i$$

Means $\forall i \in [1, n].\ x \in U_i$, then

$$\forall r \in [i, n].\ \exists r_1, ..., r_n > 0.\ B_{d(x, r_i)} \subseteq U_i$$

Now define $r := \min\{r_1, ..., r_n\} > 0$

$$\begin{aligned}
&\Longrightarrow \forall i \in [1, n].\ B_{d(x,r)} \subseteq B_{d(x,r_i)} \subseteq U_i \\
&\Longrightarrow B_{d(x,r)} \subseteq U_i \\
&\Longrightarrow B_{d(x,r)} \subseteq U_1 \cap ... \cap U_n \\
&\Longrightarrow x \text{ is an interior point of } U_1 \cap ... \cap U_n \\
&\Longrightarrow \forall x \in \bigcap_{i \in [1,n]} U_i.\ x \text{ interior} \\
&\Longrightarrow \bigcap_{i \in [1,n]} U_i \text{ is open}
\end{aligned}$$

□

1.3 Topological spaces

Definition. Let X be a set. A **topology** on a set X is a collection $\mathcal{T} \subseteq \mathcal{P}(X)$ of subsets $U \subseteq X$ called **$\mathcal{T}$-open subsets** such that
(T1) $\emptyset, X \in \mathcal{T}$
(T2) any union of members of $\mathcal{T}$ belongs to $\mathcal{T}$
(T3) any finite intersection of members of $\mathcal{T}$ belongs to $\mathcal{T}$

In this case, $(X, \mathcal{T})$ is called a **topological space**.

Example (Every metric space is a top. space)**.** Let (X, d) be a metric space. Then

$$\mathcal{T}_d := \{d\text{-open subsets of } \ X\}$$

is a topology on X.

Remark. Different metrics on X may give rise to different topologies on X.

Example (Discrete top. space)**.** Let X be any set. Then

$$\mathcal{T} := \mathcal{P}(X) = \text{Powerset of } \ X$$

is a topology on X, called the **discrete topology**, induced by the discrete metric. X with the discrete topology is called the **discrete topological space**.

Example (Indiscrete/trivial top. space)**.** Let X be any set. Then

$$\mathcal{T} := \{\emptyset, X\}$$

is a topology on X, called the **indiscrete topology**.

Definition. An open set that contains a point x is called an **open neighborhood** of x.

Definition (Hausdorff, or T_2)**.** A topological space is called **Hausdorff** if for any $x, y \in X$, $x \neq y$, there exist **disjoint** open sets $U, V \subseteq X$ such that

$$U \ni x \text{ and } V \ni y$$

Theorem. Every metric space (X, d) is Hausdorff.

proof. Let $x, y \subseteq X, x \neq y$. Then $r := d(x, y) > 0$. Now define $U := B_d(x, \frac{r}{2}), V := B_d(y, \frac{r}{2})$, meaning U, V are disjoint open neighborhood of x, y, thus X is Hausdorff. □

Theorem. If X has greater than one element, then the trivial topology on X is **not** Hausdorff.

proof. In the trivial topology, the only open neighborhood of any point $x \in X$ is X itself. So for any $x, y \in X, x \neq y$, there are no disjoint open sets $U, V \subseteq X$ such that $U \ni x$ and $V \ni y$.

□

Example. $X = \{a, b\}, a \neq b$. Possible topologies:
- $\mathcal{T}_1 = \{\emptyset, X\}$: trivial
- $\mathcal{T}_2 = \{\emptyset, \{a\}, X\}$
- $\mathcal{T}_2 = \{\emptyset, \{b\}, X\}$
- $\mathcal{T}_3 = \{\emptyset, \{a\}, \{b\}, X\}$: discrete

Example. $X = \mathbb{R}$. Define:

$$\mathcal{T} = \{\text{unions of half-open intervals of the form } [a, b) \text{ for all } a < b \in \mathbb{R}\}$$

$\mathcal{T}$ is a topology on $\mathbb{R}$, called the lower **limit topology** on $\mathbb{R}$.

<u>Notation</u>: $\mathbb{R}_{LL} = (\mathcal{T}, \mathbb{R})$

Question. How is $\mathbb{R}_{\text{LL}}$ related to $\mathbb{R}$ with the usual topology (i.e. the topology induced by the Euclidean metric)?

Answer: *They are not the same.* $[a, b)$ is open in $\mathbb{R}_{\text{LL}}$ but not with respect to the standard topology on $\mathbb{R}$.

Theorem. Every d-open subsets $U \subseteq \mathbb{R}$ is always open in $\mathbb{R}_{\text{LL}}$.

proof. Suppose $U \subseteq \mathbb{R}$ is d-open, and let $x \in U$, then x is an interior point of U with respect to d. So $\exists r > 0$. $B_d(x, r) \subseteq U$ and $U = \bigcup_{x \in U} [x, x + r)$, U is open in $\mathbb{R}_{\mathrm{LL}}$. □

Definition. X any set, $\mathcal{T}, \mathcal{T}'$ topologies on X.

- $\mathcal{T}$ is **finer** than $\mathcal{T}'$ if $\mathcal{T} \supseteq \mathcal{T}'$
- $\mathcal{T}$ is **coarser** than $\mathcal{T}'$ if $\mathcal{T} \subseteq \mathcal{T}'$

Remark. Lower limit topology on $\mathbb{R}$, $\mathbb{R}_{\mathrm{LL}}$, is finer than the standard topology on $\mathbb{R}$.

Example. $(X, \mathcal{T})$ top. space, $Y \subseteq X$.

$$\mathcal{T}|_Y := \{U \cap Y \mid U \in \mathcal{T}\}$$

is a topology on Y, called the **subspace topology induced** by $\mathcal{T}$.

Definition (subspace)**.** $(Y, \mathcal{T}|_Y)$ is called a **subspace** of $(X, \mathcal{T})$.

Theorem. If $\mathcal{T}$ is induced by a metric d on X, then the subspace topology $\mathcal{T}|_Y$ on Y is induced by the metric $d|_Y$.

1.4 Summary

Spaces in $\mathbb{R}^n$ with $d = d_{\mathrm{Eucl}}$
$\implies$ Subspaces of $\mathbb{R}^n$
$\implies$ General metric spaces
$\implies$ General topological spaces

Question. Is every metric space equivalent (as in homeomorphic) to a subspace of $\mathbb{R}^n$ for some $0 \leq n < \infty$?

Answer: *No.* We will see that any subspaces of $\mathbb{R}^n$ is 2nd countable, but e.g. $(\mathbb{R}, d_{\mathrm{discr.}})$ is not 2nd countables.

Fact (Nagata-Smirnov)**.** For all metric space (X, d), there exists a set J (very big, possibly infinite) such that (X, d) is homeomorphic to a subspace of $(\mathbb{R}^K, d_u)$.

Here:

$$\begin{aligned} \mathbb{R}^J &:= \{f : J \to \mathbb{R}\} \\ d_u &:= \text{uniform metric on } \mathbb{R}^J \\ d_{u(f,g)} &:= \sup\{\min\{1, d(f(x), g(x))\} \mid x \in J\} \end{aligned}$$

1.5 Bases for topologies

Definition. Let X be set, a collection $\mathcal{B} \subseteq \mathcal{P}(X)$ is **a base for topology on X** if

(1) $X = \bigcup_{B \in \mathcal{B}} B$

(2) If $B, B' \in \mathcal{B}$, then $B \cap B'$ is a union of members of $\mathcal{B}$.

Given such a base $\mathcal{B} \subseteq \mathcal{P}(X)$, we can define

$$\mathcal{T}_{\mathcal{B}} := \{\text{Unions of members of } \mathcal{B}\}$$

<u>Can check</u>: If $\mathcal{B}$ satisfies (1) and (2), then $\mathcal{T}_{\mathcal{B}}$ is a topology on X.

Remark. $\mathcal{T}_{\mathcal{B}}$ is the **coarsest** topology on X for which all members of $\mathcal{B}$ are open. Conversely, if a topology $\mathcal{T}$ on X is already given, then a base for $\mathcal{T}$ is collection $\mathcal{B} \subseteq \mathcal{T}$ such that every $U \in \mathcal{T}$ is a union of members of $\mathcal{B}$.

Remark. Every top. space $(X, \mathcal{T})$ has a base, namely $\mathcal{B} = \mathcal{T}$.

Example. $X = \mathbb{R}$, $\mathcal{B} = \{(a, b) \mid a < b\}$. In this case, $\mathcal{T}_{\mathcal{B}}$ is the usual topology given by $d(x, y) = |x, y|$.

Example. Let (X, d) be a metric space, $\mathcal{B} := \{B_d(x, r) \mid x \in X, r > 0\}$.

Definition (2nd countable)**.** $(X, \mathcal{T})$ is **2nd countable** if it has a **countable** base.

Example. $(\mathbb{R}^n, d = d_{\text{Eucl}})$ is 2nd countable.

$$\mathcal{B} := \{B_d(x, r) \mid x \in \mathbb{Q}^n, r \in \mathbb{Q}, r > 0\}$$

is a countable base.

Definition (Neighborhood base)**.** Let $(X, \mathcal{T})$ be top. space. A **neighborhood base** at $x \in X$ is a collection $\mathcal{N}_x \subseteq \mathcal{T}$ of $\mathcal{T}$-open neighborhoods of x such that for every $\mathcal{T}$-open neighborhood N of x, there exists $N' \in \mathcal{N}_x$ such that $N' \subseteq N$.

Definition (1st countable)**.** $(X, \mathcal{T})$ is **1st countable** if every $x \in X$ has a countable neighborhood base.

Example. Every metric space (X, d) is 1st countable.

proof. Given $x \in X$, let

$$\mathcal{N}_x := \{B_d(x, r) \mid r \in \mathbb{Q}, r > 0\}$$

and this is a countable neighborhood base at x. □

Theorem. 2nd countable implies 1st countable.

proof. Suppose $(X, \mathcal{T})$ has a countable base $\mathcal{B}$. Let $x \in X$, then

$$\mathcal{N}_x := \{B \in \mathcal{B} \mid x \in B\}$$

is a countable neighborhood base at x. $\square$

Caution. The converse is not true.

Example. $(\mathbb{R}, d = d_{\text{disc}})$ is 1st countable since $\mathcal{N}_x = \{\{x\}\}$ is a countable neighborhood base at x, but it is not 2nd countable.

Theorem. If X is an uncountable space with the discrete topology then X is not 2nd countable.

proof. X uncountable & discrete

$\Rightarrow$ Evert set in X is open
$\Rightarrow$ Evert 1-point set in X is open
$\Rightarrow$ If $\mathcal{B}$ is any base for X, then every 1-point set must be union of members of $\mathcal{B}$
$\Rightarrow$ Every 1-point set must be a member of $\mathcal{B}$
$\Rightarrow$ $\mathcal{B}$ contains uncountably many members
$\Rightarrow$ X is not 2nd countable

$\square$

Example. $X = \mathbb{R}_{\text{LL}}$ ($\mathbb{R}$ with lower limit topology) is 1st countable but not 2nd countable.

proof. $\mathcal{N}_x = \{[x, x+r) \mid n \in \mathbb{Q}^+\}$ is a countable neighborhood base at x, so X is 1st countable. But let $\mathcal{B}$ be any base for $\mathbb{R}_{\text{LL}}$. For a point $x \in \mathbb{R}$, choose[1] a base set $B_x \in \mathcal{B}$ containing x such that $B_x \subseteq [x, x+1)$. Consider the map

$$\begin{aligned} \mathbb{R} &\longrightarrow \mathcal{B} \\ x &\longmapsto B_x \end{aligned}$$

It's easy to see this map is injective because $x = \inf B_x$, implies that $|\mathcal{B}| \geq |\mathbb{R}|$, so $\mathcal{B}$ is uncountable and $\mathbb{R}_{\text{LL}}$ is not 2nd countable. $\square$

[1]May requires Axiom of Choice

1.6 Continuity

Definition (Continuity)**.** Let $(X, \mathcal{T})$, $(Y, \mathcal{T}')$ be top. spaces, $f : X \to Y$. f is **continuous** at $x \in X$ if the preimage $f^{-1}(v)$ of every $\mathcal{T}'$-open set V is $\mathcal{T}$-open. So a continuous map $f : X \to Y$ induces a map

$$\begin{aligned} \mathcal{T} &\longleftarrow \mathcal{T}' \\ f'(V) &\longleftarrow\!\shortmid V \end{aligned}$$

Example . Let X be any set, $\mathrm{id} : X \to X$ be the identity map, and $\mathcal{T}$, $\mathcal{T}'$ be two topologies on X. When is $\mathrm{id} : (X, \mathcal{T}) \to (X, \mathcal{T}')$ continuous?

$$\begin{aligned} & \mathrm{id} : (X, \mathcal{T}) \to (X, \mathcal{T}') \text{ is continuous} \\ \iff & \text{the preimage under id of each } \mathcal{T}'\text{-open set is } \mathcal{T} \text{ open} \\ \iff & \text{each } \mathcal{T}'\text{-open set is also } \mathcal{T}\text{-open} \\ \iff & \mathcal{T} \text{ is finer than } \mathcal{T}' \end{aligned}$$

Remark . The identity map of a top. space $(X, \mathcal{T})$ is always continuous.

Example . Let $(X, \mathcal{T})$, $(Y, \mathcal{T}')$ top. spaces., $y_0 \in Y$, $f : X \to Y$ the constant map, i.e., $\forall x \in X.\ f(x) := y_0$. Then f is continuous.

proof. Let $V \subseteq Y$ be any subsets, then

$$f^{-1}(V) = \begin{cases} X & \text{if } y_0 \in V \\ \emptyset & \text{if } y_0 \neq V \end{cases}$$

then the preimage of any $\mathcal{T}'$-open set is $\mathcal{T}$-open, f is continuous. □

Remark . Constant maps are always continuous. Furthermore, if X contains only one point, then any map $f : X \to Y$ is continuous.

Definition (Closed sets)**.** Let $(X, \mathcal{T})$ top. space, $A \subseteq X$ is **closed** if $X - A = X \setminus A$ is open.

Example . $X = \mathbb{R}$, $A = [0, 1]$. A is closed in $\mathbb{R}$ because

$$R - [0, 1] = (-\infty, 0) \cup (1, \infty)$$

Caution . There exist sets that are neither open nor closed. And there exist sets that are both closed and open called **clopen**. For example, $[0, 1) \subseteq \mathbb{R}_{\mathrm{LL}}$ is clopen or in any space X, the sets $\emptyset, X$ are clopen.

Definition . X is **connected** if the only clopen subsets of X are $\emptyset, X$.

Theorem. $f : X \to Y$ is continuous iff the preimage of every closed set in Y is closed in X.

proof. idea: If $A \subseteq Y$ is any subset, then

$$f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$$

so taking complements is "compatible" with taking preimages & exchanges open and closed sets. □

Theorem (Properties of closed set). $(X, \mathcal{T})$ be top. space. $Y \subseteq X$ subspace equipped with the subspace topology $\mathcal{T}|_Y := \{U \cap Y \mid U \text{ open in } X\}$. $B \subseteq Y$ is closed in $\mathcal{T}|_Y$ iff there exists a closed set $A \subseteq X$ such that

$$B = A \cap Y$$

proof. Suppose $B \subseteq Y$ is closed in $\mathcal{T}|_Y$,
$\Rightarrow$ $V := Y \setminus B$ is open in $\mathcal{T}|_Y$
$\Rightarrow$ $V = U \cap Y$ for an open set $U \subseteq X$
$\Rightarrow$ $B = Y \setminus V = Y \setminus (U \wedge Y)$

$$\begin{aligned} &= Y \setminus U \\ &= Y \cap (X \setminus U) \\ &= Y \cap A \\ &= A \cap Y \end{aligned}$$

Conversely the proof is similar. □

Remark.

1. If $Y \subseteq X$ is open in X and $V \subseteq Y$ is open in Y, then V is open in X.
2. If $Y \subseteq X$ is closed in X and and $B \subseteq Y$ is closed in $\mathcal{T}|_Y$, then B is closed in X.

Theorem. $(X, \mathcal{T})$ top. space.
1. $\emptyset, X$ are closed
2. the intersection of any collection of closed sets is closed
3. the union of any **finite** collection of closed sets is closed

proof.
1. $\emptyset$ is closed because $X - \emptyset = X$ is open, and X is closed because $X - X = \emptyset$ is open.
2. Let $A_i \subseteq X$ be closed for $i \in I$, then $\overset{X}{A_i}$ are open, $\bigcup(X \setminus A_i)$ is open, by de Morgan's law, $X \setminus \bigcap A_i$ is open, so $\bigcap A_i$ is closed.
3. Similar

□

Caution. Infinite unions of closed sets are in general **not** closed.

Example. Take $(\mathbb{R}, d)$, $A_i = \left[0, 1 - \frac{1}{i}\right]$ for $i = 1, 2, 3, \ldots$, then

$$\bigcup A_i = [0, 1)$$

which is **not** closed.

Example. X be any set, let $\mathcal{F} = \{U \subseteq X \mid X \setminus U \text{ is finite}\} \cup \{\emptyset, X\}$ defines a topology on X called the **cofinite topology** or **finite-complement topology**.

Theorem. In a Hausdorff space, every 1-point set is closed.

proof. Let X be Hausdorff, $x \in X$. For each $y \in X - \{x\}$, there exists disjoint open neighborhoods $U_x \ni x$ and $V_y \ni y$, then

$$X - \{x\} = \bigcup_{y \in X - \{x\}} V_y$$

is open, so $X - \{x\}$ is closed, meaning $\{x\}$ is closed. □

Corollary. In a Hausdorff space, every **finite** set is closed.

Corollary. if X is itself **finite**, then **every** subset of X is closed, so X is discrete.

1.7 Closure and Interior

Definition (Closure). The **closure** of $A \subseteq X$ is the set

$$\overline{A} := \bigcap \{\text{closed subsets } C \subseteq X \mid A \subseteq C\}$$

$\overline{A}$ is the smallest closed subset of X that contains $A \subseteq X$.

Definition (Interior). The **interior** of $A \subseteq X$ is the set

$$\operatorname{int} A := \bigcup \{\text{open subsets } U \subseteq X \mid U \subseteq A\}$$

$\operatorname{int} A$ is the largest open subset of X that is contained in A.

Definition (Boundary). The **boundary** of $A \subseteq X$ is the set

$$\operatorname{Bd} A := \overline{A} \cap \overline{X \setminus A}$$

Remark. By de Morgan,

$$X \setminus \overline{A} = \operatorname{int}(X \setminus A)$$
$$\overline{X \setminus A} = X \setminus \operatorname{int}(A)$$

so

$$\begin{aligned} \operatorname{Bd} A &= \overline{A} \cap (X \setminus \operatorname{int}(A)) \\ &= \overline{A} \setminus \operatorname{int}(A) \end{aligned}$$

Remark. By definition, $\operatorname{int} A \subseteq A \subseteq \overline{A}$ so A closed iff $A = \overline{A}$, and A open iff $A = \operatorname{int}(A)$. A is clopen if $A = \operatorname{int}(A) = \overline{A}$ and $\operatorname{Bd} A = \varnothing$.

Theorem. X top. space, $A \subseteq X$, $x \in X$.

1. $x \in \overline{X}$ iff every open neighborhood of x intersects A.
2. $x \in \operatorname{int} A$ iff there exists an open neighborhood of x that is contained in A.
3. $x \in \operatorname{Bd} A$ iff every open neighborhood of x intersects both A and $X \setminus A$.

proof.

1. $$\begin{aligned} x \notin \overline{A} &\Longleftrightarrow x \in X \setminus \overline{A} = \operatorname{int}(X \setminus A) \\ &\Longleftrightarrow \exists \text{ an open neighborhood of } x \text{ that's in } X \setminus A. \\ &\Longleftrightarrow \exists \text{ an open neighborhood of } x \text{ that does not intersect } A. \end{aligned}$$
2. Follows from the definition of $\operatorname{int} A$
3. Follows from 1. and from

$$\operatorname{Bd} A = \overline{A} \cap \overline{X \setminus A}$$

□

Example. $X = \mathbb{R}$ with standard top. $A = [0, 1]$, $\operatorname{int} A = (0, 1)$, $\operatorname{Bd} A = \overline{A} \cap \operatorname{int} A = \{0, 1\}$.

Example. $X = \mathbb{R}^2$ with standard top. $A = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1\} = B((0,0), 1)$. A open because $\operatorname{int} A = A$.

$$\begin{aligned} \overline{A} &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \le 1\} \\ \operatorname{Bd} A &= \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\} \\ &= S^1 = \text{unit circle} \end{aligned}$$

Fact. In any metric, $\overline{B_{d(x,r)}} \subseteq \{y \in X \mid d(x, y) \le r\}$.

Theorem. X, Y be any top. space, $f : X \to Y$ is continuous iff

$$\forall A \subseteq X.\ f\left(\overline{A}\right) \subseteq \overline{f(A)}$$

A continuous map sends points that are "extremely close" to A to points that are extremely close to $f(A)$.

Definition (Restriction). If $f : X \to Y$ is any map and $A \subseteq X$, then $f|_A$ denotes the **restriction**

$$f|_A : A \to Y = f \circ i$$

where

$$\begin{aligned} i : A &\longrightarrow X \\ x &\longmapsto x \end{aligned}$$

Fact. $i : A \to X$ is continuous with respect to the subspace topology

Lemma (Piecing lemma). Let $f : X \to Y$ is any map. Suppose $X = A \cup B$ where $A, B \subseteq X$ are closed. If $f|_A$ and $f|_B$ are continuous, then f is continuous.

proof. Need to show that the preimage of each closed set in Y is closed in X. Let $C \subseteq Y$ be closed. Then

$$\begin{aligned} f^{-1}(C) &= f^{-1}(C) \cap (A \cup B) \\ &= (f^{-1}(C) \cap A) \cup (f^{-1}(C) \cap B) \\ &= (f|_A)^{-1}(C) \cup (f|_B)^{-1}(C) \text{ is closed} \end{aligned}$$

□

Theorem. $f : X \to Y$, $g : Y \to Z$ both continuous, then so is $g \circ f : X \to Z$

proof. Let $W \subseteq Z$ be open sets, then $g^{-1}(W)$ is open and $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$ is open. □

Remark. The conclusion of the lemma also holds under the following assumptions:

- $X = \underbrace{A_1 \cup \ldots \cup A_n}_{\text{Finitely Many}}$ where all A_i are closed in X and all $f|_{A_i}$ are continuous.
- $X = \underbrace{\bigcup A_i}_{\text{Arbitrary Union}}$ where all A_i are open in X and all $f|_{A_i}$ are continuous.

In general, it does **not** hold if $X = A \cup B$ where A is open and B is closed.

1.8 Homeomorphisms

Definition (Homeomorphisms)**.** A **homeomorphism** $f : X \to Y$ is a bijection so that f and f^{-1} are both continuous.

If such f exists, we say that X and Y are **homeomorphic** and write

$$X \cong Y$$

Remark.

1. Inverses and compositions of homeomorphisms are homeomorphisms, meaning $\cong$ is an **equivalence relation** on the class of top. spaces.
2. A homeomorphism $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ induces a bijection $\mathcal{T} \iff \mathcal{T}'$, thus $X \cong Y \Rightarrow |\mathcal{T}| = |\mathcal{T}'|$
3. A property of top. space X is called a **homeomorphism invariant** or a **topological invariant** if it is preserved under $\cong$. For example, $|X|$, $|\mathcal{T}|$, Hausdorff, etc.

Example. $\mathrm{id} : (X, \mathcal{T}) \to (X, \mathcal{T})$ is a homeomorphism.

Example. If $\mathcal{T} \subseteq \mathcal{P}(X)$ is strictly finer than $\mathcal{T}' \subseteq \mathcal{P}(X)$, then

$$\mathrm{id} : (X, \mathcal{T}) \to (X, \mathcal{T}')$$

is a continuous bijection but not a homeomorphism.

Example. $X = \{a, b\}$, $a \neq b$, recall that there exists 4 topologies:

- $\mathcal{T}_1 = \{\emptyset, X\}$
- $\mathcal{T}_2 = \{\emptyset, \{a\}, X\}$
- $\mathcal{T}_3 = \{\emptyset, \{b\}, X\}$
- $\mathcal{T}_4 = \{\emptyset, \{a\}, \{b\}, X\}$

Only $\mathcal{T}_2$ and $\mathcal{T}_3$ can be homeomorphic given by

$$\begin{aligned} f : X &\longrightarrow X \\ a &\longmapsto b \\ b &\longmapsto a \end{aligned}$$

Example. Take $[0, 1]$ and $[0, 2]$ as subspaces of $\mathbb{R}$ with usual topology. They are homeomorphic by the map

$$\begin{aligned} f : [0, 1] &\longrightarrow [0, 2] \\ x &\longmapsto 2x \\ f^{-1} : [0, 1] &\longleftarrow [0, 2] \\ \frac{y}{2} &\longleftarrow\!\shortmid y \end{aligned}$$

Likewise, $S^1 \cong 2S^1$ where S^1 is the unit circle.

Example. $(0,1) \cong \mathbb{R}$ given by homeomorphism

$$f(x) := \tan\left(\pi x - \frac{\pi}{2}\right)$$
$$f^{-1}(y) := \frac{1}{\pi}\left(\arctan(y) + \frac{\pi}{2}\right)$$

Example. $f : [0,1) \to S^1$ given by

$$f(x) := (\cos(2\pi x), \sin(2\pi x)) = e^{2\pi i x}$$

is a continuous bijection, but f^{-1} is not continuous: at $(1,0) \in S^1$, f^{-1} does not satisfy the ε-δ condition. In fact, $[0,1] \ncong S^1$.

Theorem (Piecing lemma for homeomorphisms)**.** $X = A \cup B$, $Y = C \cup D$, $A, B \subseteq X$ closed, $C, D \subseteq Y$ closed. $f : X \to Y$ a map $f(A) = C$, $f(B) = D$. Suppose f is a bijection and $f|_A : A \to C$ and $f|_B : B \to D$ are homeomorphisms, then f is a homeomorphism.

Remark (Construction of homeomorphisms)**.** Let $J \subseteq [0, 2\pi]$ be subset, suppose $g_1, g_2 : J \to [a,b]$, where $0 < a < b < \infty$ are continuous functions. D_1, D_2 are the subsets of $\mathbb{R}^2$ given by

$$D_i = \left\{(r,\theta) \mid \theta \in J \text{ and } 0 \leq r \leq g_{i(\theta)}\right\}$$

in polar coordinates. Claim: $D_1 \cong D_2$. Idea is to define a homeomorphism $f : D_1 \to D_2$ by sending each radial segments in D_1 linearly to the corresponding radial segment in D_2. Put Formally:

$$f(r,\theta) := \left(\frac{g_2(\theta)}{g_1(\theta)} r, \theta\right)$$

can check that f is a homeomorphism.

Example. $X = D^2 = \{(x_1, x_2) \subseteq \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}$, $Y = [-1,1] \times [-1,1]$, $X \cong Y$. or,

$$X = \{x \in \mathbb{R}^2 \mid \|x\|_{\text{Eucl}} \leq 1\}$$
$$Y = \{x \in \mathbb{R}^2 \mid \|x\|_{\text{Max}} \leq 1\}$$

define $f : X \to Y$ by

$$f(x) := \begin{cases} 0 & \text{if } x = 0 \\ \frac{\|x\|_{\text{Eucl}}}{\|x\|_{\text{Max}}} x & \text{otherwise} \end{cases}$$

Same works in $\mathbb{R}^n$:

$$\{x \leq \mathbb{R}^n \mid \|x\|_{\text{Eucl}} \leq 1\} \cong [-1,1]^n$$

Definition (Isometry) **.** Let $(X, d), (Y, d')$ be metric spaces. Any **isometry** $f : X \to Y$ is a bijection so that

$$\forall x_1, x_2 \in X.\ d'(f(x_1), f(x_2)) = d(x_1, x_2)$$

Remark . **Isometries** are injective and continuous, and every bijective isometry is a homeomorphism.

Example (Some isometries of $\mathbb{R}^2$)**.**
- Rotations by an angle φ
- Reflections along lines
- Translations
- Glide reflections

Claim . $f : \mathbb{R}^2 \to \mathbb{R}^2$ any map, the following are equivalent:
(1) f is an isometry fixing 0
(2) $\forall x, y \in \mathbb{R}^2.\ \langle f(x), f(y)\rangle = \langle x, y\rangle$
(3) $f(x) = Ax$ for an orthogonal ($A^t A = I_2$) matrix A

And 3 implies that such map is linear

proof.
- (3) ⇒ (1) Let $f(x) = Ax$ for A orthogonal, then f fixes 0 and

$$\begin{aligned} d(f(x), y(x)^2 &= d(Ax, At)^2 \\ &= \langle Ax - Ay,\ Ax - Ay\rangle \\ &= \langle A(x-y),\ A(x-y)\rangle \\ &= (x-y)^t A^t A(x-y) \\ &= (x-y)^t(x-y) \\ &= d(x,y)^2 \end{aligned}$$

- (1) ⇒ (2) Let f be an isometry fixing 0. Follows because

$$\langle x, y\rangle = \frac{1}{2}(d(x,y)^2 - d(x,0)^2 - d(y,0)^2)$$

- (2) ⇒ (3) Suppose f preserves $\langle\ ,\ \rangle$. Let $A = (a_1\ a_2)$ where $a_1 := f(e_1), a_2 := f(e_2)$, then a_1, a_2 are orthogonal, so A is orthogonal. Let $h(x) := A^t f(x) = A^{-1} f(x)$. Then h preserves $\langle\ ,\ \rangle$ and fixes e_1, e_2, so

$$\begin{aligned} h(x) &= \langle h(x), e_1\rangle e_1 + \langle h(x), e_2\rangle e_2 \\ &= \langle h(x), h(e_1)\rangle e_1 + \langle h(x), h(e_2)\rangle e_2 \\ &= \langle x, e_1\rangle e_1 + \langle x, e_2\rangle e_2 = x \\ \Rightarrow h(x) &= x \\ \Rightarrow A^{-1} f(x) &= x \Rightarrow f(x) = Ax \end{aligned}$$

□

1.9 Linear and affine maps

Easy to see: Every linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is continuous and every invertible linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is homeomorphism.

Definition. $f : \mathbb{R}^2 \to \mathbb{R}^2$ is **affine** if

$$\forall a_1, a_2 \in \mathbb{R}^2, \lambda_1, \lambda_2 \in \mathbb{R}.\ f(\lambda_1 a_1 + \lambda_2 a_2) = \lambda_1 f(a_1) + \lambda_2 f(a_2) \text{ with } \lambda_1 + \lambda_2 = 1$$

Exercise. In this case

$$f\left(\sum_{i=1}^{n} \lambda_i a_i\right) = \sum_{i=1}^{n} \lambda_i f(a_i)$$

with $\sum_{i=1}^{n} \lambda_i = 1$.

Claim. f affine and fixes 0 iff f is linear.

proof.

- "$\Leftarrow$" obvious
- "$\Rightarrow$" Suppose f is affine and fixes 0, and let $\lambda_1, \lambda_2 \in \mathbb{R}$ be arbitrary. Then

$$\begin{aligned} f(\lambda_1 a_1 + \lambda_2 a_2) &= f(\lambda_1 a_1 + \lambda_2 a_2 + (1 - \lambda_1 - \lambda_2)0) \\ &= \lambda_1 f(a_1) + \lambda_2 f(a_2) + (1 - \lambda_1 - \lambda_2) f(0) \\ &= \lambda_1 f(a_1) + \lambda_2 f(a_2) \\ &\Rightarrow f \text{ is linear} \end{aligned}$$

$\square$

Remark. Any constant map is affine, and linear combinations of affine maps are affine.

Corollary. Every affine $f : \mathbb{R}^2 \to \mathbb{R}^2$ has the form

$$f(x) = Ax + b$$

fir a 2×2 matrix A and $v \in \mathbb{R}^2$. Invertible affine maps $f : \mathbb{R}^2 \to \mathbb{R}^2$ are homeomorphisms.

Remark (Special case)**.** Let $\lambda \in \mathbb{R}$, $\lambda \neq 1, \lambda \neq 0$, $f(x) = \lambda x + v$ is a **deletion** or **scaling** by λ with fixed point

$$\frac{1}{1 - \lambda} v$$

Definition (Affinely independent). $a_1, a_2, a_3 \in \mathbb{R}^2$ are **affinely independent** if

$$\forall \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}.\ \lambda_1 a_1, \lambda_2 a_2, \lambda_3 a_3 = 0 \quad \Longrightarrow \quad \lambda_1 = \lambda_2 = \lambda_3 = 0$$

Easy to see, a_1, a_2, a_3 affinely independent iff $a_2 - a_1, a_3 - a_1$ linearly independent. Geometry: a_1, a_2, a_3 are not collinear.

Fact . If $a_1, a_2, a_3 \in \mathbb{R}^2$ are affinely independent and $b_1, b_2, b_3 \in \mathbb{R}^2$ are arbitrary, then there exists a unique affine map $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$f(a_i) = b_i \text{ for } i = 1, 2, 3$$

Definition . A homeomorphism composed by multiple maps with piecing lemma is called a PL homeomorphism, where PL stands for **piecewise linear**.

Definition (triangulation). A **triangulation** for $\mathbb{R}^2$ is a collection T of triangles $t \in \mathbb{R}^2$ such that

1. the $t \in T$ cover $\mathbb{R}^2$
2. if two $t \neq t' \in T$ meet, then $t \cap t'$ is either a common edge or a common vertex.
3. Every bounded set $B \subseteq \mathbb{R}^2$ meets only finitely many $t \in T$.

Here,

triangle Euclidean triangle, non-degenerate, and the interior is non-empty

bounded fits into a $B(a, r) \in \mathbb{R}^2$ for r sufficiently large

Definition . A bijection $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a **PL homeomorphism** if there exists triangulations T, T' of $\mathbb{R}^2$ such that f maps each $t \in T$ affinely (and bijectively) to a $t' \in T'$.

Fact . Every PL homeomorphism of $\mathbb{R}^2$ is a homeomorphism.

proof. First use the piecing lemma (for finite unions of closed sets) to show that f is continuous on each bounded set $B \subseteq \mathbb{R}^2$. Then use the piecing lemma (for arbitrary union of open sets) to conclude that f is continuous on all of

$$\mathbb{R}^2 = \bigcup_{r>0} B(a, r)$$

Finally, repeat the argument to conclude that f^{-1} is also continuous. □

Remark (Some types of homeo's of $\mathbb{R}^2$ with d_{Eucl}).

Type	**Algebraic Description**	**Examples**
Isometries	$f(x) = Ax + v \quad \mathbb{R}^2 \rtimes O(2)$	congruent triangles
Isometries & scaling	$\mathbb{R}^2 \rtimes (O(2) \times \mathbb{R}^2)$	similar triangles
Affine bijection	$\mathbb{R}^2 \rtimes GL_2(\mathbb{R})$	any triangles
PL homeomorphisms	–	simple polygons

Where:
$\mathbb{R}^2$ Additive group of $\mathbb{R}^2$
$\mathbb{R}^+$ Additive group of strictly positive real numbers
$\boldsymbol{O(2)}$ set of orthogonal 2×2 matrices
$\boldsymbol{GL_2(\mathbb{R})}$ set of real invertible 2×2 matrices

1.10 Topological Properties

Properties of a topological space $(X, \mathcal{T})$ that are preserved under homeomorphisms:
- $|X|$ (number of points)
- $|\mathcal{T}|$ (number of open sets)
- Minimal cardinality of a basis or a neighborhood base

2nd countable has a countable base
1st countable every point has a countable neighborhood base

1.10.1. Separation Properties

Definition (Regular). X is **regular** if it is Hausdorff and for all closed $C \subseteq X$ and $x \in X \setminus C$ there exist disjoint open sets $U, V \subseteq X$ such that $C \subseteq U$ and $x \in V$

Definition (Normal). X is **normal** if it is Hausdorff and for all disjoint closed $C, D \subseteq X$ there exist disjoint open sets $U, V \subseteq X$ such that $C \subseteq U$ and $D \subseteq V$

Remark. Normal $\implies$ regular $\implies$ Hausdorff

Theorem. A Hausdorff space X is normal iff there exists $U \supseteq C$ and every open neighborhood $U \subseteq C$ there exists an open neighborhood $V \supseteq C$ such that $\overline{V} \subseteq U$

Theorem. A Hausdorff space X is normal iff for all decomposition $X = U \cup V$ into open sets $U, V \subseteq X$, there exists open sets $U', V' \subseteq X$ such that $X = U' \cup V'$ and $\overline{U'} \subseteq U$ and $\overline{V'} \subseteq V$.

Theorem. Every metric space X, d is normal *proof.* Already seen: (X, d) is Hausdorff. To show it's normal, Let $C_1, C_2 \subseteq X$ be disjoint closed sets. For each $x \in C$, let

$$r_x := d(x, C_2) := \inf\{d(x, y) \mid y \in C_2\} > 0$$

for each $y \in C_2$, let

$$r_y := d(y, C_1) := \inf\{d(y, x) \mid x \in C_1\} > 0$$

define:

$$U := \bigcup_{\{x \in C_1\}} B_{d(x, \frac{r_x}{2})}, V := \bigcup_{\{y \in C_2\}} B_{d\left(y, \frac{r_y}{2}\right)}$$

and one can check $U \cap V = \emptyset$. Thus X is normal. □

1.11 Compactness

Definition (Open Cover)**.** X top. space, $A \subseteq X$ subset. An **open cover** of A is a collection of open sets $U_1 \subseteq X$ such that

$$\bigcup U_i \supseteq A$$

Definition (Subcover)**.** A **subcover** of U is a subcollection $V \subseteq U$ which is still a cover of A

Definition. $A \subseteq X$ is **compact** if every open cover of A has a **finite** subcover.

Special case: $A = X$. X is compact if every open cover of X has a finite subcover.

So: compactness can be seen as a property for

- A top. space X
- A subset $A \subseteq X$

Easy to see: $A \subseteq X$ compact $\iff$ A compact as a top. space equipped with the subspace topology
Usually: regard compactness as a property for top. spaces

Remark. Compactness is preserved under homeomorphisms.

Theorem. Let $f : X \to Y$ be continuous, if X is compact, then so is $f(X) \subseteq Y$

proof. Let $V = \{V_i\}$ be an open cover of $f(X) \subseteq Y$, $U := \{f^{-1}(V_i)\}$ is an open cover of X since f is continuous. Since X is compact, there exists a finite subcover $U_{i_1}, \ldots, U_{i_k} \in U$ of X, then $V_{i_1}, \ldots V_{i_k}$ is a finite subcover of V_i. Thus $f(X)$ is compact. □

Example. $\mathbb{R}$ is not compact.

$$U := \{(-r, r) \mid r > 0\}$$

is an open cover with no finite subcover.

Example. X metric space, $A \subseteq X$ an unbounded subset, then A is not compact.

proof. Fix $x \in X$,

$$\bigcup_{r>0} B_{d(x,r)} = X \supseteq A$$

is an cover of A with no finite subcover. □

Remark. In a metric space, compact subsets must be bounded. In fact, they must be totally bounded, i.e., for every $\varepsilon > 0$, they can be covered by finitely many ε-balls.

Example. $(0, 1)$ us **not** compact, since $(0, 1) \cong \mathbb{R}$

Example. $[0, 1]$ and $[0, 1]^n$ are compact

Example. If a topology space X has only finitely many open sets, it is compact.

Theorem. In a Hausdorff space, every compact subset is closed.

proof. Let X be Hausdorff and $A \subseteq X$ be compact.

<u>Need to show</u>: Every $x \in X \setminus A$ is an interior point of $X \setminus A$.

Consider $y \in A$, since X is Hausdorff, there exists disjoint open neighborhoods $U_y \ni x$ and $V_y \ni y$, then $\{V_y \mid y \in A\}$ is an open cover for A. Since A is compact, there exists a finite subcover $V_{y_1}, \ldots, V_{y_n}$. Now define

$$U := U_{y_1} \cap \ldots \cap U_{y_n}$$

then U is an open neighborhood of x and

$$\begin{aligned} U \cap A &\subseteq U \cap \left(\bigcup V_{y_i}\right) \\ &= \bigcup \left(U \cap V_{y_i}\right) \\ &= \emptyset \\ \implies Y &\subseteq X \setminus A \end{aligned}$$

hence x is an interior point of $X \setminus A$. Thus $X \setminus A$ is open, meaning A is closed. □

Theorem. In a compact space, every closed subset is compact.

proof. Let X be compact and $A \subseteq X$ be closed. Let $U = \{U_i\}$ be an open cover of A. Then $X \setminus A$ is open, so $U \cup \{X \setminus A\}$ is an open cover of X. Since X is compact, there exists a finite subcover $U_{i_1}, \ldots, U_{i_k}, X \setminus A$. Then $U_{i_1}, \ldots, U_{i_k}$ is a finite subcover of A. $\square$

Definition. A map $f : X \to Y$ is open (resp., closed) if the image of each open (resp., closed) subset of X is open (resp., closed) in Y.

Example. The map $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x_1, x_2) := x_1$, is **open** because the image of any ball is opened interval.

Caution. f is not closed. Let

$$A := \left\{ (x_1, x_2) \in \mathbb{R} \mid x_1 \neq 0 \text{ and } x_2 = \frac{1}{x_1} \right\}$$

A is closed in $\mathbb{R}^2$ but $f(A) = \mathbb{R} - \{0\}$ is not closed in $\mathbb{R}$.

Remark. If $f : X \to Y$ is a bijection, f closed iff f open iff f^{-1} continuous.

Theorem (Compact-to-Hausdorff Theorem)**.** Let $f : X \to Y$ be a continuous map from a compact space X to a Hausdorff space Y. Then $f(X)$ is compact.

proof. Let $C \subseteq X$ be closed, then C is compact, $f(C)$ is compact, and $f(C)$ is closed. Thus $f(X)$ is compact. $\square$

Corollary. Let $f : X \to Y$ be a continuous bijection from a compact space X to a Hausdorff space Y. Then f is a homeomorphism.

proof. f is a closed map, so f^{-1} is continuous. $\square$

Caution. The assumption on X and Y are essential!

Example. $f : [0, 1) \to S^1$ given by $f(s) = (\cos(2\pi s), \sin(2\pi s))$ is a continuous bijection but not a homeomorphism.

Example. $X = \{a, b\}, a \neq b$. Equip X with the discrete topology, and $Y = \{a, b\}$, but with trivial topology.

$$\mathrm{id} : X \longrightarrow Y$$

is a continuous bijection, but not a homeomorphism.

1.12 Compactness in $\mathbb{R}$

Fact. Every non-empty bounded above subset $A \subseteq \mathbb{R}$ has a least upper bound in $\mathbb{R}$

Theorem. $[a, b] \in \mathbb{R}$ is compact

Corollary. $A \subseteq \mathbb{R}$ compact iff A closed and bounded.

1.13 Product Topology and compactness in $\mathbb{R}^n$

Let X, Y be topological spaces.

$$\mathcal{B} := \{U \times V \mid U \subseteq X \text{ open in } U, V \subseteq Y \text{ open in } Y\}$$

Easy to see: this is a basis for a topology on $X \times Y$ denoted $\mathcal{T}_{\text{product}}$.

Definition (Product neighborhood). A $U \times V$ that contains $(x, y) \in X \times Y$ is called a **product neighborhood** of (x, y).

Definition (Two possible extensions to infinite product). $X_i, i \in I$, family of top, space, let X be the set $\prod_{i \in I} X$

1.

$$\mathcal{B}_{\text{product}} := \left\{ \prod_{i \in I} U_i \mid U_i \text{ open in } X_i, U_i = X_i \text{ for all but finitely many } i \right\}$$

Product Topology on $X = \prod X_i$

2.

$$\mathcal{B}_{\text{box}} := \left\{ \prod_{i \in I} U_i \mid U_i \text{ open in } X_i \right\}$$

Box Topology on $X = \prod X_i$

For finer topologies, $\mathcal{T}_{\text{box}}$ are usually finer than $\mathcal{T}_{\text{product}}$

Proposition. Let $X \times Y$ be equipped with $\mathcal{T}_{\text{product}}$, then the inclusion $i_X : Y \to X \times Y$, $i_Y : X \to Y$, the projections $p_X : X \times Y \to X$, $p_Y : X \times Y \to Y$ are continuous and open.

Remark. $\mathcal{T}_{\text{product}}$ is the smallest (coarsest) topology on $X \times Y$ that makes the projections continuous.

Corollary. For fixed $x \in X$ and $y \in Y$, $\{x\} \times Y$ and $X \times \{y\}$ are homeomorphic to Y and X, respectively.

Theorem (Tychonoff's Theorem). Let $X_i, i \in I$ be a family of compact top. spaces. Then the product $\prod_{i \in I} X_i$ is compact.

proof. We will need tube lemma.

Lemma (Tube lemma). Let Y be compact and $W \subseteq X \times Y$ be an open neighborhood of $\{x\} \times Y$ for an $x \in X$, then there exists an open neighborhood $U \subseteq X$ of x such that $U \times Y \subseteq W$. $U \times Y$ is sometimes called a "tube".

□

Claim. The standard topology on $\mathbb{R}^n$, $0 \leq n < \infty$ agrees with the product topology on

$$\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}$$

Theorem (Heine-Borel). $A \subseteq \mathbb{R}^n$ compact iff A is closed and bounded.

Fact (Generalized Heine-Borel). (X, d) be metric space. X compact iff X complete (every Cauchy sequence has a convergent subsequence) and totally bounded.

Remark. "Compact" is a topological property. "Complete" and "totally bounded" are metric properties that are preserved under bijective isometries.

Definition (Sequences). Let X be topological space. A **sequence** in X is a function

$$s : \mathbb{N} \to X$$

usually write: $s_n := s(n)$ and $\{s_n \mid n \in \mathbb{N}\}$ or $\{s_1, s_2, \ldots\}$ for the sequences.

Definition (Subsequence). A **subsequence** of a sequence s in X is a composition $s' = s \circ j$ for a strictly increasing function $j : \mathbb{N} \to \mathbb{N}$.

<u>In other hands</u>: A subsequence is of the form

$$\{s_{j_1}, s_{j_2}, s_{j_3}, \ldots\}$$

for $j_1 < j_2 < j_3 < \ldots$

Definition (Convergence). s be a sequence in X, x be any point in X. We say that s **converges** to x if for every open neighborhood U of x, there exists $N \in \mathbb{N}$ such that $s_n \in U$ for all $n \geq N$.

Equivalently, every open neighborhood of x contains s_n for all but finitely many n.

Remark. In Hausdorff space, a sequence s can converge to at most one point. In this case, we say that x is the **limit** of s and write

$$x = \lim_{n \to \infty} s_n$$

Definition (Sequentially Compactness). A topological space X is **sequentially compact** if every sequence in X has a convergent subsequence.

Question. How is this related to compactness?

<u>Preliminary observation</u>: If $s_n \to x$, then every open neighborhood U of x contains s_n for infinitely many n.

Proposition. Let $x \in X$, if X is 1st countable and if every open neighborhood of x contains s_n for infinitely many n, then s_n has a convergent subsequence converging to x.

proof. Since X is 1st countable, there exists a countable neighborhood basis

$$\mathcal{N}_x = \{N_1, N_2, N_3, \dots\}$$

at x. Define

$$M_i := N_1 \cap \dots \cap N_i$$

then $\{M_1, M_2, \dots\}$ is a new neighborhood basis at x and

$$M_1 \supseteq M_2 \supseteq M_3 \supseteq \dots$$

In particular, every open neighborhood U of x contains all M_i with $i \gg 0$. Then we can choose $j_1 < j_2 < j_3 < \dots$ such that

$$s_{j_i} \in M_i$$

then $\left\{s_{j_1}, s_{j_2}, s_{j_3}, \dots\right\}$ is a subsequence of s that converges to x. □

Theorem. If X is 1st countable, then X compact $\Rightarrow$ X sequentially compact.

proof. Suppose X is compact, and suppose S_n is a sequence in X that has no convergent subsequence, then $\forall x \in X$. $\{s_n\}$ has no subsequence converging to x. Since X is comact, there exists finitely many $x_1, \dots, x_m \in X$ such that $U_{x_1} \cup \dots \cup U_{x_m} = X$, then X contains s_n for only finitely many n, contradiction because $\{s_n\}$ was a sequence in X. □

1.14 Lebesgue number lemma

Definition (Lebesgue number)**.** Let (X, d) be metric space, $\mathcal{U} = \{U_i\}$ be an open cover for X. A real number $\mathcal{S} > 0$ is called a **Lebesgue number** for $\mathcal{U}$ if every $A \subseteq X$ with $\varnothing(A) < \mathcal{S}$ is contained in some $U_i \in \mathcal{U}$.

Theorem (The Lemma)**.** If a metric space X is sequentially compact then every open cover of X has a Lebesgue number.

proof. Let X be sequentially compact and $\mathcal{U} = \{U_i\}$ be an open cover. Suppose there is no Lebesgue number for $\mathcal{U}$, there there are arbitrarily small $A \subseteq X$ which are not in U_i. Then there exists a sequence $A_1, A_2, \ldots \subseteq X$ such that

$$\varnothing(A_n) < \frac{1}{n}$$

but since that A_n is not contained in any U_i, choose $a_n \in A_n$ in each A_n, get a sequence $a_1, a_2, \ldots$ in X that has no convergent subsequence, contradiction. □

Theorem. If X is a metric space, then X sequentially compact $\Rightarrow$ X compact.

1.15 Connectness

Definition (Separation)**.** A **separation** for top. space X is a pair of disjoint non-empty open subsets $U, V \subseteq X$ such that $X = U \cup V$

Definition. X is **separated** or **disconnected** if there exists a separation for X. Otherwise X is **connected**.

Note. If $X = U \cup V$ is a separation, then U and V are clopen in X.

Remark. X connected
$\iff$ X is not a disjoint union of two non-empty closed subsets
$\iff$ the only clopen subsets of X are $\varnothing$ and X

Definition . A subset $A \subseteq X$ is **separated** (resp., **connected**) if A is separated (resp., connected) in the subspace topology.

Note. Connectness is a topological property.

Theorem. Let $f : X \to Y$ be continuous. If X is connected, then so is $f(X)$.

1.16 Connectness in $\mathbb{R}$

Definition. $A \subseteq \mathbb{R}$ is **convex** if

$$x, y \in A \Rightarrow [x, y] \subseteq A$$

Remark. Convexity also makes sense in $\mathbb{R}^n$.

Lemma. $A \subseteq \mathbb{R}$ connected $\Rightarrow$ $A \subseteq \mathbb{R}$ convex, but not true in general.

proof. Suppose $A \subseteq \mathbb{R}$ is connected but not convex, then

$$\begin{gathered} \exists x, y \in A.\ [x, y] \notin A \\ \Rightarrow \exists z \in [x, y].\ z \notin A \\ \Rightarrow (-\infty, z) \cap A \text{ and } (z, \infty) \cap A \text{ is a separation for } A \\ \Rightarrow A \text{ is disconnected} \\ \bot \end{gathered}$$

□

Lemma. $A \subseteq \mathbb{R}$ connected $\Rightarrow$ $A \subseteq \mathbb{R}$ is an interval, a ray, or $\mathbb{R}$.

proof. Assume for simplicity that A is bounded. Let $a := \inf A$ and $b := \sup A$, then $[a, b] \subseteq A$.

<u>Exercise</u>: Since A is convex, $A \supseteq (a, b)$

4 possibilities: $A = [a, b]$, $A = [a, b)$, $A = (a, b]$, $A = (a, b)$, then A is an interval □

Lemma. $A \subseteq \mathbb{R}$ an interval, a ray, or $\mathbb{R}$ $\Rightarrow$ $A \subseteq \mathbb{R}$ connected.

proof. Wil only consider the case

$$A = [a, b] \text{ for } a < b$$

Suppose $U, V \subseteq \mathbb{R}$ are open subsets such that $U \cap A$ and $V \cap A$ are disjoint and $U \cup V \supseteq A$.

<u>Need to show</u>: $U \supseteq A$ or $V \supseteq A$.

Assume WLOG that $a \in U$, define

$$B := \{x \in [a, b] \mid [a, x] \subseteq U\}$$

Notice that $a \in B$ since $a \in U$, then $B \neq \emptyset$ and B is bounded since $\subseteq [a, b]$. Let

$$u := \sup B \geq a$$

Now prove as an exercise: $u \in U$, $u \in B$, $u = b$, then $b \in B$, hence $B = [a, b] \subseteq U$, thus $A \subseteq U$. □

Summary. The following are equivalent:

(a) A connected
(b) A convex
(c) A an interval, a ray, or $\mathbb{R}$

Theorem (IVT)**.** Let $f : X \to \mathbb{R}$ be continuous. If X is connected and f assumes two values $x, y \in \mathbb{R}$, then it also assumes every value $z \in [x, y]$.

proof. The assumptions imply that $f(X) \subseteq \mathbb{R}$ is connected, then $f(X)$ convex. □

Corollary (EVT)**.** Let $X \neq \emptyset$ be connected and compact, then

$$f(X) = [m, M]$$

where m is the absolute minimum of f and M is the absolute maximum of f.

1.16.1. Application

Theorem. Let $f : S^1 \to \mathbb{R}$ be continuous. Then

$$\exists x \in S^1.\ f(x) = f(-x)$$

proof. Define $g : S^1 \to \mathbb{R}$ by

$$g(x) := f(x) - f(-x) \in \mathbb{R}$$

then g is continuous and $g(-x) = -g(x)$. Now fix $x \in S^1$. Let $\alpha \in S^1$ be one of the arcs from x to $-x$. Let $k := g|_\alpha : \alpha \to \mathbb{R}$. Can assume WLOG $k(x) \geq 0$, then $k(-x) \leq 0$, hence there exists $y \in \alpha$ such that $k(y) = 0$, then $g(y) = 0$, and $f(y) = f(-y)$. □

Remark. Turns out: If $n \geq 1$ and $f : S^n \to \mathbb{R}^n$ is continuous, then

$$\exists x \in S^n.\ f(-x) = f(x)$$

1.17 Path Connectness

Definition (Path)**.** X top. space, $x, y \in X$, a **path** from x to y is a continuous map $f : [0, 1] \to X$ such that

$$f(0) = x \text{ and } f(1) = y$$

In this case, say f connects x to y.

Definition (Path Connectness)**.** X is **path connected** if

$$\forall x, y \in X.\ \exists \text{ a path } f \text{ from } x \text{ to } y$$

Remark. if f is a path from x to y, then

$$\overline{f} := f(1-t), t \in [0,1]$$

is a path from y to x.

Remark. Can also compose paths in X:

f is a path from x to y, g is a path from y to z, define

$$(f * g)(t) := \begin{cases} f(2t) & 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

is a path from x to z

Definition (Path Component)**.** Define a relation $\sim$ on X by

$$x \sim y := \exists \text{ a path } f \text{ from } x \text{ to } y$$

$\sim$ is an equivalence relation, and the equivalence classes are called **path components**, turns out, the maximal path connected subset of X.

Definition. Define a in general different relation $\sim$ on X by

$$x \sim y :\iff \exists \text{ a connected } A \subseteq X.\ A. \ni x, y$$

$\sim$ is an equivalence relation, and the equivalence classes are called **connected components** or just **components**, turns out, the maximal connected subset of X.

Theorem. X path connected $\Rightarrow$ X connected

proof. Suppose X is path connected but not connected, then there exists a separation $U, V \subseteq X$. Let $x \in U$, $y \in V$, there exists a path f from x to y and $[0,1] = f^{-1}(U) \cup f^{-1}(V)$ is a separation for $[0,1]$, contradiction. $\square$

Remark (Consequence)**.** Each path component of X is connected
$\Rightarrow$ each path component is in a connected component
$\Rightarrow$ each connected compoenet is a disjoint union of path components

Example. Let $X = A \cup B \subseteq \mathbb{R}^2$,

$$A := \left\{ \left(x, \sin\left(\frac{1}{x}\right) \right) \mid x \in (0,1] \right\}$$
$$B := \{(0,y) \mid y \in [-1,1]\} = \{0\} \times [-1,1]$$

Equip $X \subseteq \mathbb{R}^2$ with the subpace topology from $\mathbb{R}^2$. Then X is connected but not path connected. This is called the **topologist's sine curve**.

Fact. Connected components are closed, but path components are not necessarily closed.

Proposition. Every convex subset $A \subseteq \mathbb{R}^n$ is path connected

proof. Obvious: if $x, y \in A$, then

$$g(t) := (1-t)x + ty, t \in [0,1]$$

is a path in A from x to y. □

Definition. X is **locally path connected** if for every $x \in X$ and every open neighborhood U of x, there exists an open neighborhood $V \subseteq U$ of x that is a neighborhood basis of x consisting of path connected sets.

Remark. This is equivalent to saying that X has a basis consisting of path connected sets.

Example. $\mathbb{R}^n$ is locally path connected

Proposition. Suppose X is locally path connected, and $U \subseteq X$ is open in X, then

$$U \text{ connected} \Longrightarrow U \text{ path connected}$$

while $\Longleftarrow$ always holds.

Definition (Quotient Spaces)**.** $q : X \to Y$ be surjective maps, then

$$U \subseteq Y \text{ open} \Longleftrightarrow q^{-1}(U) \subseteq X \text{ open}$$

In this case:
- Y is called the **quotient space** of X by g
- Say Y has the quotient topology w.r.t. (X, Q)

where $\mathcal{T}_{\text{quotient}} = \{U \subseteq Y \mid q^{-1}(U) \subseteq X \text{ open}\}$

Corollary. Suppose

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"] & X' \arrow[d, "q'"] \\ Y \arrow[r, "g"] & Y' \end{tikzcd}$$

is a commutative diagram of continuous maps, where q is a quotient map. If f is continuous, then so is g.

1.18 Construction of quotient spaces

Definition. X space, $\sim$ equivalence relaton on X, consider the map

$$q : X \to \frac{X}{\sim}, \quad \text{all} \sim \text{equivalence classes } [x]$$

can equip $\frac{X}{\sim}$ with the quotient topology,

$$U \subseteq \frac{X}{\sim} \text{ open} \iff q^{-1}(U) \subseteq X \text{ open}$$

$\left(\frac{X}{\sim}, \tau_{\text{quotient}}\right)$ is called an **identification space**.

Example (Special Case)**.** $A \subseteq X$ subset, define

$$x \sim y :\iff x = y \text{ or } x, y \in A$$

Definition.

$$\frac{X}{A} := \frac{X}{\sim}$$

"collapsed" A to a single point.

Proposition. Let $f : X \to Y$ be a continuous surjection from a compact space X to a Hausdorff space Y. Define

$$\forall x, x' \in X.\ x \sim x' \iff f(x) = f(x')$$

Then the induced map $\overline{f} : \frac{X}{\sim} \to Y$ is a homeomorphism.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\text{quotient map: } q"'] & Y \\
\frac{X}{\sim} \arrow[ur, "\overline{f}"'] &
\end{tikzcd}$$

proof. It is clear that $\overline{f}$ is a continuous bijection. Moreover, $\frac{X}{\sim}$ is compact since $\frac{X}{\sim} = g(X)$. Then $\overline{f}$ is a continuous bijection from a compact space to a Hausdorff space, thus a homeomorphism. $\square$

Example. $X = [0,1]$, $A = \{0,1\}$

Claim: $\frac{[0,1]}{\{0,1\}} = \frac{[0,1]}{0\sim 1} \cong S^1$

proof. Define $f : [0,1] \to S^1$ by

$$f(t) := e^{2\pi i t}$$

then f is continuous and surjective. Because $[0,1]$ is compact, S^1 Hausdorff, then $f(t) = f(t') \iff t = t'$ or $t, t' \in [0,1]$. Therefore $\overline{f} : \frac{[0,1]}{\{0,1\}} \to S^1$ is a homeomorphism. $\square$

Example.

$$X = D^n = \{(x_1, ..., x_n) \in \mathbb{R}^2 \mid x_1^2 + ... + x_n^2 \leq 1\} \subseteq \mathbb{R}^n$$
$$A = \partial D^n = S^{n-1} = \{... \mid x_1^2 + ... + x_n^2 = 1\}$$

Claim: $\frac{D^n}{\partial D^n} \cong S^n$

proof. Define $f : D^n \to S^n$ by

$$f(x) := \left(\underbrace{\sin(\pi \|x\|) \frac{x}{\|x\|}}_{\in \mathbb{R}^n}, \underbrace{-\cos(\pi\|x\|)}_{\in \mathbb{R}} \right) \in \mathbb{R}^n \times \mathbb{R}$$

for $x = 0$ and $f(0) := (0, ..., 0, -1)$

f continuous injection. D^n compact, S^n Hausdorff, $f(x) = f(x')$ iff $x = x'$ or $x, x' \in \partial D^n$, then $\overline{f} : \frac{D^n}{\partial D^n} \to S^n$ is a homeomorphism. $\square$

1.19 Disjoint Union and gluing

Definition (Disjoint Union Topology). A, B disjoint topological spaces, if not, could make them disjoint by replacing them by $A \times [0]$ and $B \times \{1\} \in (A \cup B) \times \{0, 1\}$ as a set. Define

$$\mathcal{T} := \{U \cup V \mid U \text{ open in } A, V \text{ open in } B\}$$

then $\mathcal{T}$ is a topology on $A \cup B$ called the **disjoint union topology**. Use the notion

$$A \sqcup B := (A \cup B, \mathcal{T})$$

Note. A, B are clopen in $A \sqcup B$, then if $A, B \neq \emptyset$, then $A \sqcup B$ is disconnected.

Definition. Suppose $K \subseteq B$ is a subset, $f : K \to A$ a continuous map (or homeomorphism)[2], assume K is closed in B (and K closed in A), then define

$$A \sqcup_f B := \frac{A \sqcup B}{f(x) \sim x, \forall x \in K}$$

Why do wee want K to be closed and f to be continuous?

Claim. The subspace topology on $A \subseteq A \sqcup_f B$ agree with the original topology on A.

proof. Clear that the quotient map

$$q : A \sqcup B \to A \sqcup_f B$$

restrict to a continuousjection from

$$A \xrightarrow{\text{id}} A$$

from A with original topology to A as a subspace of $A \sqcup_f B$. For every closed set $C \subseteq A$ in subspace topology is also closed in the original topology.

$$q^{-1}(C) = C \cup f^{-1}(C)$$

where $f^{-1}(C)$ is closed in K since f continuous, hence closed in B. Then $q^{-1}(C)$ is closed in $A \sqcup B$, then C is closed in $A \sqcup_f B$ by defintion of $\mathcal{T}_{\text{quotient}}$. □

Proposition. Let

$$g : A \sqcup_f B \longrightarrow C$$

be induced by continuous maps

$$g_A : A \longrightarrow C \text{ and } g_B : B \longrightarrow C$$

[2]Text in grey parenthesis are extra assumptions in textbook

such that $g_A \circ f = g_B|_K$ then g is continuous.

proof. Have

$$\begin{tikzcd} A \sqcup B \arrow[r, "g_A \sqcup g_B"] \arrow[d, "q"'] & C \\ A \sqcup_f B \arrow[ur, "g"'] & \end{tikzcd}$$

g continuous by the universal property of quotient topology.. □

Lemma (Urysohn's)**.** Let A, B be disjoint closed subsets in a normal space X, then there exists a continuous $f : X \to [0, 1]$ s.t.

$$f(A) \subseteq \{0\} \text{ and } f(B) \subseteq \{1\}$$

proof. Slightly lengthy (non-obvious) in general. For metric space, can take:

$$f(x) := \frac{d(x, A)}{d(x, A) + d(x, B)}$$

where $d(x, A) := \inf\{d(x, y) \mid y \in A\}$ □

Remark. Could replace $[0, 1]$ by any $[a, b]$

Remark. If $f : X \to [0, 1]$ is a Urysohn function for A and B, then

$$U := f^{-1}\left(\left[0, \frac{1}{2}\right)\right), V := f^{-1}\left(\left(\frac{1}{2}, 1\right]\right)$$

are disjoint open neighborhoods for A and B.

Theorem (Tietze Extension Theorem)**.** Let X normal, $A \subseteq X$ closed, $f : A \longrightarrow [a, b]$ continuous, $a \leq b$. Then there exists a continuous $F : X \to [a, b]$ such that $F|_A = f$.

Lemma. The Tietze Extension Theorem also holds for continuous function

$$f : A \longrightarrow \mathbb{R}$$

for $A \subseteq X$ closed and X normal.

1.20 Simply connected space

Recall. X path connected if $\forall x, y \in X.$ there exists a path from x to y. Equivalently, every (cont.) map $g : S^0 \to X$ extends to a continuous map $G : D^1 \to X$, where $D^1 = [-1, 1], S^0 = \partial D^1 = \{-1, 1\}$.

Definition. X simply connected if it is path connected and every continuous map $g : S^1 \to X$ extends to a continuous map $D^2 \to X$.

Fact.
1. Convex subsets $A \subseteq \mathbb{R}^n$ are simply connected
2. S^1 or $\mathbb{R}^2 - \{(0, 0)\}$ or $\mathbb{R}^3 - \{\text{z axis}\}$ are not simply connected
3. If $X = U \cup V$ where
   - $U, V \subseteq X$ are open
   - U, V are simply connected
   - $U \cap V$ is path connected

   then X is simply connected

Example. $X = S^2$ or S^n for $n > -2$, $N = (0, 0, 1)$, north pole, $S = (0, 0, -1)$, south pole. Define

$$U := S^2 - \{N\} \cong \mathbb{R}^2$$
$$V := S^2 - \{S\} \cong \mathbb{R}^2$$

then U, V are open and simply connected, $U \cup V = S^2$. Moreover, $U \cap V = S^2 - \{N, S\}$ is path connected, then S^2 is simply connected.

1.21 Jordan Curve Theorem and Schoenflies Theorem

Definition. A **simply closed curve** in $\mathbb{R}^2$ is a continuous injection $f : S^1 \to \mathbb{R}^2$.

Remark. The image $C = f(S^1)$ is sometimes also called a “simple closed curve”

Theorem (Jordan Curve Theorem). If $C = f(S^1)$ is a simple closed curve in $\mathbb{R}^2$, then $\mathbb{R}^2 \setminus C$ has exactly two connected components. Moreover:
- one these components is bounded and the other one is unbounded
- C is the boundary of each of these components

There exists various proofs, e.g. via (co)homology (Lefschetz duality). Direct proof:

proof. Can compactify $\mathbb{R}^2$ to get $\mathbb{R}^2 \cup \{\infty\} \cong S^2$. Map C onto S^2 without touching ∞. Now remove $p \in C \subseteq S^2$. to get an "infinite arc" $C' \subseteq \mathbb{R}^2$. This reduces the original problem to showing:

Lemma . If $f : \mathbb{R} \to \mathbb{R}^2$ is a closed embedding and $C' = f(\mathbb{R})$, then $\mathbb{R}^2 \setminus C'$ is not path connected.

proof. Let $C' = f(\mathbb{R})$, $f : \mathbb{R} \to \mathbb{R}^2$ be a closed embedding. Let $g : C' \xrightarrow{\cong} \mathbb{R}$ be the homeomorphism which is inverse of f. By Tietze Extension Theorem, g extends to a continuous $G : \mathbb{R}^2 \to \mathbb{R}$. Think $\mathbb{R}^3$ as the product of xy-plane and z-axis. Define $F : \mathbb{R}^3 \to \mathbb{R}^3$ as the composition

$$\begin{gathered}(p, z) \longmapsto (p, z + G(p)) \longmapsto (p - f(z + G(p)), z + G(p)) \\ (p, z') \longmapsto (p - f(z'), z')\end{gathered}$$

it's easy to see that F is a homeomorphism and maps $C' \subseteq \mathbb{R}^2 \subseteq \mathbb{R}^3$ bijectively to the z-axis. Then

$$\mathbb{R}^3 \setminus C' \underset{F}{\cong} \mathbb{R}^3 \setminus \{\text{z axis}\}$$

then $\mathbb{R}^3 \setminus C'$ is not simply connected. On the other hand, can write $\mathbb{R}^3 \setminus C'$ as $\mathbb{R}^3 \setminus C' = U \cup V$ where

$$\begin{gathered}U := (\mathbb{R}^2 \times (0, \infty)) \cup ((\mathbb{R}^2 \setminus C') \times (-1, 1)) \\ U := (\mathbb{R}^2 \times (-\infty, 0)) \cup ((\mathbb{R}^2 \setminus C') \times (-1, 1))\end{gathered}$$

can check that U and V are open and simply connected. Note that $U \cap V = (\mathbb{R}^2 \setminus C') \times (-1, 1)$. If $\mathbb{R}^2 \setminus C'$ were path connected, then $U \cap V$ would be path connected, contradiction, hence $\mathbb{R}^2 \setminus C'$ is not path connected. □

Note (Why is this proof nice?).
- To prove that $\mathbb{R}^2 \setminus C'$ is not path connected, would like to straiten C'
- This is hard to do in $\mathbb{R}^2$, but easier in $\mathbb{R}^3$
- <u>Turns out</u>: C' can be straightened in $\mathbb{R}^2$

□

Theorem (Schoenflies Theorem). If $C = f(S^1)$ is a simple closed curve in $\mathbb{R}^2$, then there exists a homeomorphism from $\mathbb{R}^2$ to itself which takes C to S^1.

In particular:
- This homeomorphism maps the bounded component of $\mathbb{R}^2 \setminus C$ to $D^2 \setminus \partial D^2$, meaning this component is $\cong D^2 \setminus \partial D^2$
- The unbounded componenet is $\cong \mathbb{R}^2 \setminus D^2$

proof. Not too hard if C is a simple polygon, but hard in general. Proof omitted. □

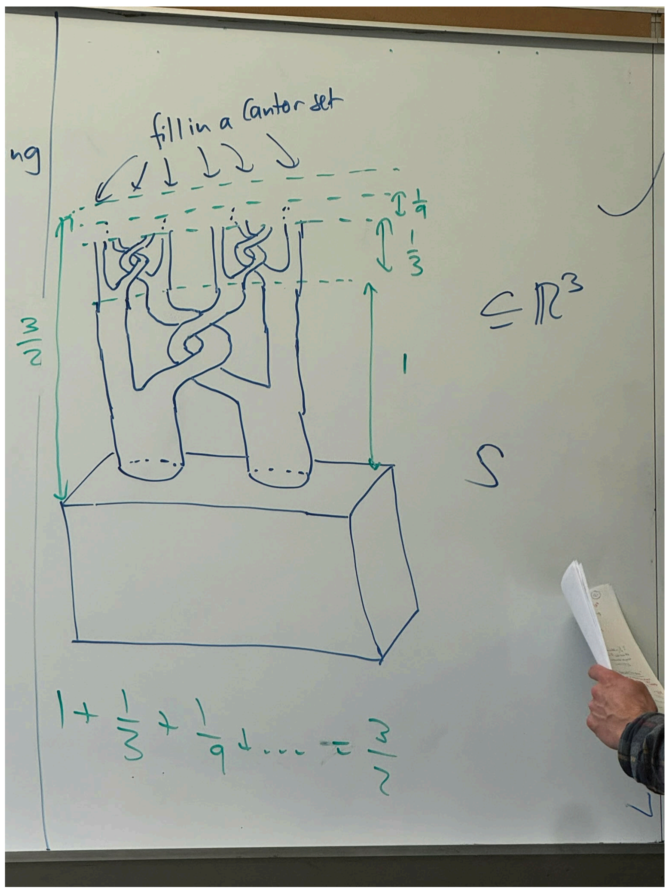
Note. Suppose $f : S^1 \to \mathbb{R}^2$ is smooth (a C^∞ diffeomorphism onto its image) and $C := f(S^1)$. Let $x \in \mathbb{R}^2 \setminus C$. How can we tell whether x is in the bounded or the unbounded component of $\mathbb{R}^2 \setminus C$?

- Choose a base point $x_0 \in \mathbb{R}^2 \setminus C$ that is "far away" from C
- Choose a smooth path $\gamma_x \subseteq \mathbb{R}^2$ for x to x_0 which intersects C transversely
- x is in the bounded component if $|C \cap \gamma_x|$ is odd and in the unbounded component otherwise.

1.21.1. Situation in higher dimensions

Let $f : S^{n-1} \to \mathbb{R}^n$ continuous injection. $S := f(S^{n-1}) \subseteq \mathbb{R}^n$. Jordan Curve Theorem remains true: $\mathbb{R}^n \setminus C$ has exactly 2 path components, can be proved by cohomology and Lefschetz duality.
But, in general:
- The bounded component is **not** $\cong D^n \setminus \partial D^n$
- The unbounded component is **not** $\cong \mathbb{R}^n \setminus D^n$

Example (in $\mathbb{R}^3$)**.** Construct a nontrivial embedding of $\mathbb{S}^2$ into $\mathbb{R}^3$.



Get an embedding of S^2 into $\mathbb{R}^3$ such that the unbounded component is not $\cong \mathbb{R}^3 \setminus D^3$.

1.22 Local flatness and collar neighborhoods

Definition (Local Flatness or Topological flatness). A topological embedding $f : S^{n-1} \to \mathbb{R}^n$ is locally **flat** if

$$\forall p \in S := f(S^{n-1}).\ \exists \text{ an open neighborhood } U \subseteq \mathbb{R}^n.\ (U, U \cap S) \cong (\mathbb{R}^n, \mathbb{R}^{n-1} \times \{0\})$$

Example. The Alexander horned sphere is not locally flat.

Example (for codimension 2 embeddings). Embed $\mathbb{R}$ into $\mathbb{R}^3$ by infinite many decreasing "knots" that has a limit point. This embedding is not locally flat.

Example (for codimension 2 embeddings). Let $K \subset \mathbb{R}^3 \subseteq \mathbb{R}^3 \cup \{\infty\} \cong S^3$ a knotted simple closed curve. In D^4, connect each point of K to the center of D^4 using a straight line segment.

Definition (bi-collared). Let $f : S^{n-1} \to \mathbb{R}^n$ is a continuous injection, $S := f(S^{n-1})$. S is **bi-collared** if there exists an open neighborhood $U \subseteq \mathbb{R}^n$ of S and a homeomorphism

$$F : (S^{n-1} \times (-1, 1)) \xrightarrow{\cong} U$$

such that $F\ |_{S^{n-1} \times \{0\}} = f$

Theorem (Brown, 1961). S locally flat $\Longrightarrow$ S bi-collared

Theorem (Generalized Schoenflies Theorem, Brown 1960). S bi-collared $\Longrightarrow$ the components of $S^n \setminus S$ are $\cong D^n \setminus \partial D^n$

Note. For $n = 2$, C is locally flat $\overset{\text{Brown}}{\Longrightarrow}$ C bi-collared $\overset{\text{GST}}{\Longrightarrow}$ the components of $S^2 \setminus C$ are $\cong D^2 \setminus \partial D^2$

Chapter 2

The classification of surfaces

Definition (Notations).

$$\begin{aligned} D^n &:= \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_1^2 + ... + x_n^2 \leq 1\} \\ S^{n-1} &:= \partial D^n \\ D^1 &= [-1, 1] \\ D^0 &= \{1 \text{ point}\} \end{aligned}$$

2.1 Manifolds

Definition (n-manifold)**.** An n-manifold is a 2nd countable Hausdorff space M such that every $x \in M$ has an open neighborhood $U \subseteq M$ with $U \cong \mathbb{R}^n$.

So, M locally "looks like" $\mathbb{R}^n$.

Fact. Every n-manifold can be embedded into $\mathbb{R}^{2n+1}$, i.e., it's homeomorphic to a subspace of $\mathbb{R}^{2n+1}$.

Theorem. Every compact n-manifold M can be embedded into $\mathbb{R}^N$ for some $N < \infty$.

proof. Cover M by finitely many open sets $U_i, ..., U_k$ with $U_i \cong \mathbb{R}^n$, possible since M is a compact n-manifold. For each i, let

$$f_i : U_i \to \mathbb{R}^n$$

be a homeomorphism. Define

$$g_i : M \to \mathbb{R}^n \cup \{\infty\} \cong S^n$$

by

$$g_{i(x)} := \begin{cases} f_i(x) & x \in U_i \\ \infty & x \notin U_i \end{cases}$$

Exercise: check that g_i is continuous (use that $U \subseteq \mathbb{R}^n\{\infty\} \subseteq S^n$ is open iff U is an open subset of $\mathbb{R}^n$ or $\infty \in U$ and $(\mathbb{R}^n \cup \{\infty\}) \setminus U$ is a compact subspace of $\mathbb{R}^n$)

Let $h_i : M \to \mathbb{R}^{n+1}$ be the composition

$$M \xrightarrow{g_i} S^n \hookrightarrow \mathbb{R}^{n+1}$$

then h_i is continuous. Now define $F : M \to \overbrace{\mathbb{R}^{n+1} \times ... \times \mathbb{R}^{n+1}}^{k \text{ copies}} = \mathbb{R}^{k(n+1)}$ by

$$F(x) := \left(h_1(x), ..., h_{k(x)}\right)$$

Exercise: F continuous injective, thus a homeomorphism onto its image because M compact and $\mathbb{R}^{k(n+1)}$ Hausdorff. □

Definition (surface)**.** A 2-manifold is called a **surface**.

Definition (n-manifold with boundary)**.** A n-manifold with boundary is a 2nd countable Hausdorff space M s.t. $\forall x \in M.\ \exists$ an open neighborhood $U \subseteq M$ of x. and a homeomorphism h from U to an open subset of $\mathbb{H}^n$, where

$$\mathbb{H}^n := \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$$
$$\partial\mathbb{H}^n := \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_n = 0\}$$

Fact.
1. If x is an interior point of M, then it has an open neighborhood $\cong \mathbb{R}^n$.
2. If x is a boundary point of M, then it has an open neighborhood $\cong \mathbb{H}^n$
3. A point $x \in \partial H$ can't simultaneously be an interior point and a boundary point.

Definition. $\partial M := \{\text{all boundary points of } M\}$

Fact. ∂M is an $(n-1)$-manifold without boundary

Definition. An n-manifold M is called **closed** if it is compact and has empty boundary.

Example. $\mathbb{R}^n$ is an n-manifold without boundary (but not closed for $n > 0$ because not compact)

Example. $\mathbb{H}^n$ is an n-manifold with boundary

Example. S^n is a closed n-manifold and $x \in S^n$ is one of the sets

$$U := S^n - \{N\} \cong \mathbb{R}^n$$
$$V := S^n - \{S\} \cong \mathbb{R}^n$$

Example . Every countable discrete space is a 0-manifold (since 2nd countable, Hausdorff, and locally homeomorphic to $\mathbb{R}^0 = \{0\}$)

Example . With $n = 1$, the only compact nonempty connected 1-manifolds are $[0, 1]$ and S^1.

Example .

$$\mathbb{H}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0 \text{ and } x_2 \geq 0\}$$

and $\mathbb{H}^2_+ \cong \mathbb{H}^2$, hence a 2-manifold with boundary

Example . Goal: Classify compact surfaces with boundary up to homeomorphism.

1. 2-Sphere

$$S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$$

2. Real Projective Plane

$$P = \mathbb{RP}^2 = \{\text{Unoriented straight lines through the origin} \in \mathbb{R}^3$$
$$= \frac{S^2}{x \sim -x}$$

Can check that the quotient map $S^2 \to P$ is a local homeomorphism (in fact, a covering map), that is, every $x \in S^2$ has a neighborhood that gets mapped homeomorphically to an open set in P.

3. Torus

$$T = S^1 \times S^1 = \frac{[0,1]^2}{\left\{ \substack{(0,t)\sim(1,t) \\ (s,0)\sim(s,1)} \right\}} \cong \frac{\mathbb{R}^2}{\mathbb{Z}^2}$$

Here, $\frac{\mathbb{R}^2}{\mathbb{Z}^2}$ means that we identify two points $x, y \in \mathbb{R}^2$ if $x - y \in \mathbb{Z}^2$. Explicit quotient:

$$\mathbb{R}^2 \longrightarrow S^1 \times S^1 = T$$
$$(s, t) \longmapsto \left(e^{2\pi i s, 2\pi i t}\right) \in \mathbb{C} \times \mathbb{C}$$

this map is a local homeomorphism.

4. Klein Bottle

$$K = \frac{[0,1]^2}{\left\{ \substack{(0,t)\sim(1,t) \\ (s,0)\sim(1-s,1)} \right\}}$$

There exists a map

$$T \longrightarrow K$$

given by

$$
\begin{aligned}
(s,t) &\longmapsto (s,2t) \quad \text{if } t \in \left[0,\frac{1}{2}\right] \\
(s,t) &\longmapsto (1-s,2t-1) \quad \text{if } t \in \left[\frac{1}{2},1\right]
\end{aligned}
$$

2.2 Invariance of domain

Theorem. $U \subseteq \mathbb{R}^n$ open. If $f : U \to \mathbb{R}^n$ is a continuous injection, then f is open.

proof. Suffices to show that every sufficiently small open ball $B(x,\varepsilon) \subseteq \mathbb{R}^n$ with $B(x,\varepsilon) \subseteq U$ is sent to an open subset of $\mathbb{R}^n$. Let B be such a ball. By making ε smaller, we can assume $\overline{B} \subseteq U$, then by Jordan Separation Theorem, $\mathbb{R}^n \setminus f(\partial B)$ has 2 path components. Moreover, $f(B)$ is path-connected since B.

Fact. $\mathbb{R}^n \setminus f\left(\overline{B}\right)$ is also path-connected

then $f(B)$ and $\mathbb{R}^n \setminus f\left(\overline{B}\right)$ must be the path componenets of $\mathbb{R}^n \setminus f(\partial B)$, then $f(B)$ is open and $\mathbb{R}^n$ is locally path connected. $\square$

Corollary. If $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ are nonempty open subsets with $U \cong V$, then $m = n$.

proof. Suppose $m \neq n$, and assume WLOG $m > n$. Consider a homeomorphism

$$f : U \xrightarrow{\cong} V$$

and compose f with the embedding

$$\mathbb{R}^n \longrightarrow \mathbb{R}^n \times \{0\} \hookrightarrow \mathbb{R}^m$$

to get a continuous injection

$$f' : U \longrightarrow \mathbb{R}^m$$

by theorem, f' is open, then $f'(U)$ is open but $f'(U) \notin \varnothing$ and $f'(U) \subseteq \mathbb{R}^n \times \{0\} \cong \mathbb{R}^n$, contradiction. $\square$

2.3 Surfaces with boundary

Definition. Let M be connected surface, possibly with boundary.
$D \subseteq M \setminus \partial M$ embedded closed disk, i.e.
$D = f(D^2)$ for a continuous injection $f : D^2 \to M \setminus \partial M$. Can check that $f(D^2 \setminus \partial D^2) = D \setminus \partial D$. Then

$$M_{(1)} := M \setminus f(D^2 \setminus \partial D^2)$$

Remark. $M_{(1)}$ is independent of the choice of D, up to homeomorphism.
Reason: M connected space, $D_1, D_2 \subseteq M \setminus \partial M$ embedded closed disks.

Lemma (Disk lemma)**.** There exists a homeomorphism with $h(D_1) = D_2$

Definition (Generalization)**.** M be connected space,

$$D_1, ..., D_n \subseteq M \setminus \partial M$$

be disjoint embedded closed disks. then

$$M_{(n)} := M \setminus \left(\bigcup_{i=1}^{n} \operatorname{int}(D_i) \right)$$

Example (Surfaces with $\partial M \neq \emptyset$)**.**

1. Closed disk

$$D_2 \cong S^2 \setminus \operatorname{int}(\text{upper hemisphere}) \cong S^2_{(1)}$$

2. Annulus / cylinder

$$\begin{aligned} S^1 \times [0,1] &\cong \frac{[0,1]^2}{(0,t) \cong (1,t)} \\ &\cong S^2 \setminus \operatorname{nbhd}\{N,S\} \\ &\cong S^2_{(2)} \\ &\cong D^2_{(1)} \end{aligned}$$

3. Möbius band

$$\frac{[0,1]^2}{(0,t) \cong (1,1-t)}$$

There exists a 2-1 map fron the annulus to the Möbius band given by

$$(s,t) \longmapsto (2s,t) \quad \text{if } s \in \left[0, \frac{1}{2}\right]$$
$$(s,t) \longmapsto (2s-1, 1-t) \quad \text{if } s \in \left[\frac{1}{2}, 1\right]$$

Also:

$$\text{Möbius band} \cong \frac{S^2 - \operatorname{nbhd}\{N,S\}}{x \sim -x} = P_{(1d)}$$

Definition (Handle)**.** An i-handle (of dimension $i+j$) is a space $D^i \times D^j$

Note . Abstractly:

$$D^i \times D^j \cong D^{i+j}$$

but the product structure on $D^i \times D^j$ will matter. For $i+j=2$:

- 0-handle: $D^0 \times D^2$ disk
- 1-handle: $D^1 \times D^1$ square
- 2-handle: $D^2 \times D^0$ disks

Definition . A 2 dimensional 2-handle body is a topology space M that is built out of 2-dimensional handles as follows:

0. Start with a finite collection of disjoint 2-dimension 0-handles

$$M_0 = \bigcup_{i=1}^{k_0} h_i^0 \quad \text{(0-handles)}$$

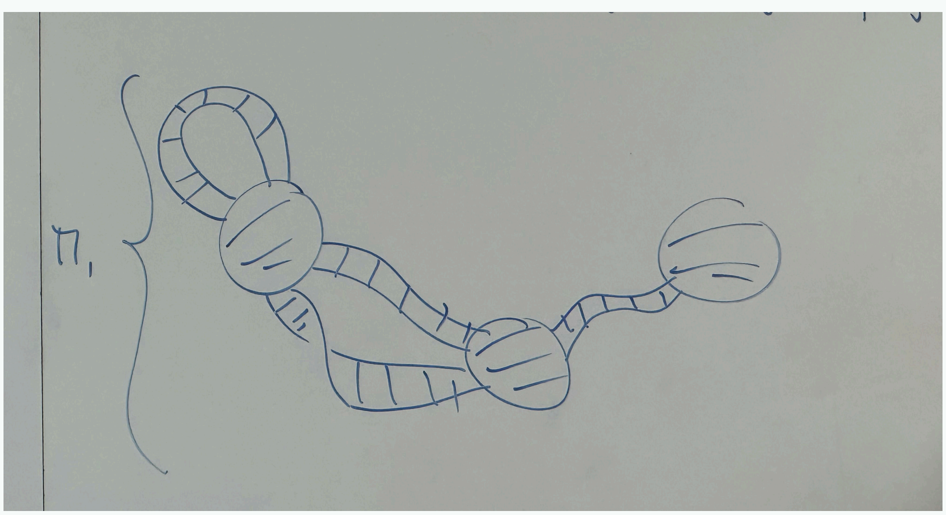
1. Build M_1 by attaching 2-dimension 1-handles to M_0. That is,

$$M_1 = M_0 \cup \bigcup_{j=1}^{k_1} h_j^1$$

where we attach h_j^1 to M_0 using a continuous attaching maps

$$f_j^1 := (\partial D^1) \times D^1 \longrightarrow \partial M_0 = \bigcup_{i=1}^{k_0} \partial h_i^0$$

we'll assume:
- f_j^1 is a topological embedding
- the images of the f_j^1 are disjoint for $j = 1, \ldots, k_1$



2. Build $M = M_2$ by attaching 2-dimensional 2-handles $h_1^2, \ldots, h_{k_2}^2$ to ∂M_1 using attaching map

$$f_j^2 : (\partial D^2) \times D^0 \longrightarrow \partial M_1$$

with the same assumptions as above.

So that M_2 is obtained from M_1 from gluing 2-disks to some boundary components of M_1.

Theorem (Rado, 1940s). Up to homeomorphism,

$$\{\text{compact surfaces with boundary}\} = \{\text{2-dimensional 2-handle bodies}\}$$

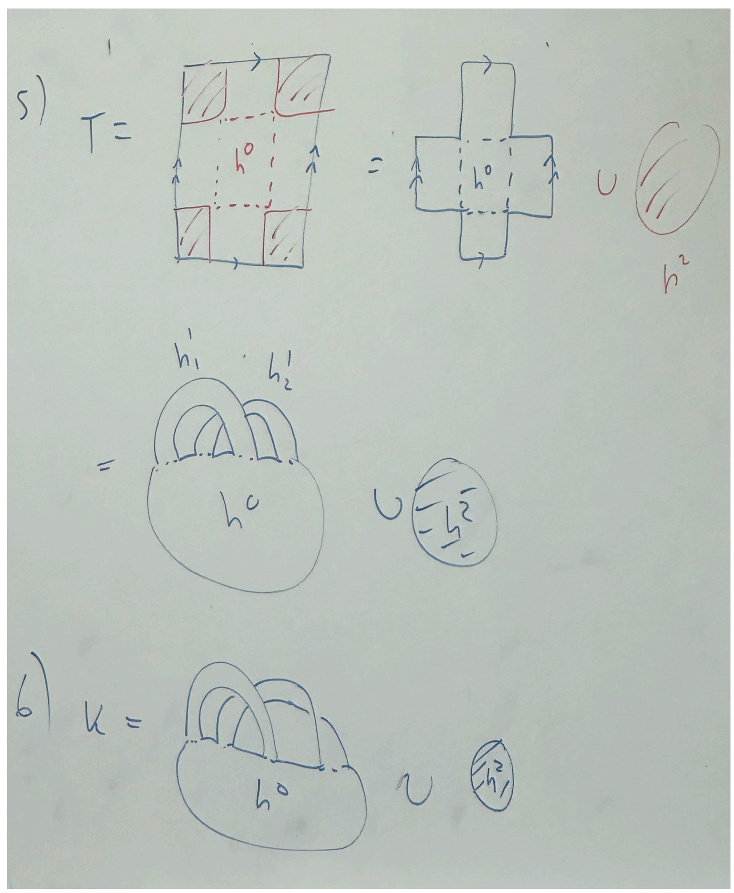
We'll use this without proof.

Definition (Handle Decomposition). An identification of a surface M with a 2-handle body is called a **handle decomposition** of M.

Example.

1. Annulus $= h^0 \cup h^1$
2. Möbius band $= h^0 \cup h^1$
3. $S^2 = h^0 \cup h^2$
4. $P_{(1)}$ = Möbius band, so

$$P = h^0 \cup h^1 \cup h^2$$



Theorem. M, H be spaces, $B \subseteq M$, $A \subseteq H$ be closed subsets, and homeomorphisms

$$f, g : A \to B$$

If $g^{-1} \circ g$ extends to a homeomorphism of H or $f \circ g^{-1}$ extends to a homeomorphism of M, then

$$H \sqcup_f H \cong M \sqcup_B M$$

proof. Assume $g^{-1} \circ f$ extends to a homeomorphism $F : H \to H$, then

$$\begin{tikzcd} M \arrow[r, "\mathrm{id}"] \arrow[d, tail, "\sqcup_f"] & M \arrow[d, tail, "\sqcup_g"] \\ H \arrow[r, "F"] & H' \end{tikzcd}$$

is the desired homeomorphism. If $f \circ g^{-1}$ extends over M, replace f and g by their inverses. $\square$

Theorem. If M is connected, then there exists only one way to attach (via an embedding $f : \partial D^2 \to \partial M$) a 2-handle up to homeomorphism.

proof. Let $f, g : \partial D^2 \to \partial M$ be two attaching maps. First note that $f(\partial D^2)$ and $g(\partial D^2)$ must be connected components S_1^1 and S_2^1 of ∂M. Suppose $S_1^1 \neq S_2^1$. By attaching disks to S_1^1, S_2^1, we get a surface $\widehat{M}$ such that $M = M^2_{(2)}$. By lemma, there exists a homeomorphism $h : M \to M$ with $h(S_1^1) = S_2^1$, then g and $h^{-1} \circ g$ differ by h, which is defined on all of M, then we can assume $S_1^1 = S_2^1$. Now f, g are both homeomorphism:

$$f, g : \partial D^2 \longrightarrow S_1^1$$

follows so is $g^{-1} \circ f$ and it extends to a homeomorphism $F : D^2 \to D^2$, explicitly,

$$F(x) := \begin{cases} 0 & \text{if } x = 0 \\ \|x\| \ (g^{-1} \circ f)\left(\frac{x}{\|x\|}\right) & \text{if } x \neq 0 \end{cases}$$

By the previous theorem,

$$M \sqcup_f D^2 \cong M \sqcup_g D^2$$

$\square$

Proposition. If M is a 2-dimensional 2-handle body, then

$$M \text{ connected} \iff M_1 (M \text{ without the 2-handles}) \text{ connected}$$

proof.
($\Longrightarrow$) Follows since every 2-handle gets attached to a single connected component of M_1.
($\Longleftarrow$) Follows because each 2-handle is connected $\square$

Remark. To clasify connected 2-handle bodies, it suffices to classify 2-handle bodies with $\partial M \neq \emptyset$.

$$\begin{array}{ccc} \dfrac{\left\{\begin{array}{c}\text{closed nonempty}\\ \text{connected surfaces}\end{array}\right\}}{\text{homeomorphism}} & \longleftrightarrow & \dfrac{\left\{\begin{array}{c}\text{compact connected}\\ \text{surfaces with exactly}\\ \text{one boundary component}\end{array}\right\}}{\text{homeomorphism}} \\ M & \longrightarrow & M_{(1)} \end{array}$$

Theorem. There exists only one way to attach a 0-handle

proof. Attaching a 0-handle is the same as taking the disjoin union with a disk D^2. $\square$

2.4 Isotopies

Definition (Isotopy)**.** Let B a space, let

$$g_0, g_1 : B \to B$$

be homeomorphisms. They are **isotopic** if there exists a continuous map

$$G : B \times I \to B$$

such that
1) $G_0 = g_0$
2) $G_1 = g_1$
3) $G_t : B \to B$ is a homeomorphism for all $t \in I$

where $G_{t(b)} := G(b, t)$. Can regard $\{G_t \mid t \in I\}$ is a "continuous family" of homeomorphisms $G_t : B \to B$. Call G an **isotopy** from g_0 to g_1.

Definition (Ambient isotopic)**.** g_0, g_1 are **ambient isotopic** if there exists an isotopy G such that

$$G_0 = \mathrm{id}_B \text{ and } G_1 \circ g_0 = g_1$$

Definition. Let $G : B \times I \to B$ be an isotopy, define

$$\tilde{G} : B \times I \longrightarrow B \times I$$

by

$$\tilde{G}(b,t) := (G(b,t),t)$$

then $\tilde{G}$ is a continuous bijection.

Fact. If B is compact and Hausdorff, then $\tilde{G}$ is a homeomorphism. (by compact-to-Hausdorff theorem)

Remark. This remains true if B is only locally compact. (idea: replace B by its 1-point compactification)

Theorem. Let M be compact surface, $h^1 = D^1 \times D^1$, the 2-dimensional 1-handle,

$$f, g : (\partial D^1) \times D^1 \longrightarrow \partial M$$

be embeddings. If f, g are ambient isotopic, then

$$M \sqcup_f h^1 \cong M \sqcup_g h^1$$

proof. We need:

Fact (Brown)**.** M be a compact surface. ∂M has a **collar neighborhood** in M. That is, a closed set $C \subseteq M$ with $C \supseteq \partial M$ and such that there exists a homeomorphism

$$\varphi : C \longrightarrow (\partial M) \times I$$

which restricts to the "identity map"

$$\partial M \longrightarrow (\partial M) \times \{1\}$$

In this case, $\varphi^{-1}((\partial M) \times (0,1])$ is open in M.

Suppose $f, g : (\partial D^1) \times D^1 \to \partial M$ are ambient isotopic, and let

$$G : (\partial M) \times I \to \partial M$$

be an ambient isotopy between f and g. Because ∂M is compact, $\tilde{G}$ is a homeomorphism. Regard $\tilde{G}$ as

$$\tilde{G} : C \longrightarrow C$$

where C is a collar neighborhood of ∂M in M. Define the homeomorphism between $M \sqcup_f h'$ and $M \sqcup_g h'$ by letting it to be id on h' and $M \setminus \varphi^{-1}(\partial M) \times (0,1]$, and $\tilde{G}$ for the rest. $\square$

2.4.1. Homeomorphisms of $I = [0, 1]$

Definition. $\mathrm{Homeo}(X) = \{\text{homeomorphism } f : X \to X\}$ is a group w.r.t. composition.

Lemma. $\mathrm{Homeo}([0, 1]) = \{\text{strictly monotone bijection } f : [0, 1] \to [0, 1]\}$

proof. Every $f \in \mathrm{Homeo}([0, 1])$ is monotonous by the intermediate value theorem. (exercise).

Conversely, if $f : [0, 1] \to [0, 1]$ is a monotonous bijection then it bijectively send intervals of the form $(a, b), [0, b), (a, 1]$, for $0 < a < b < 1$ to intervals of the same type. f is a homeomorphism because intervals form a basis for the topology of $[0, 1]$. □

Note. If $f \in \mathrm{Homeo}([0, 1])$, then
- f increasing, then f fixes 0 and 1
- f decreasing, then f swaps 0 and 1

Lemma. If $f \in \mathrm{Homeo}([0, 1])$ is increasing, then it isotopic to $\mathrm{id}_{[0,1]}$.

proof. Define

$$G_{t(s)} := (1 - t)f(s) + ts \text{ for } (s, t) \in [0, 1]^2$$

then $G_0 = f$ and $G_1 = \mathrm{id}_{[0,1]}$. Moreover, each $G_t : [0, 1] \to [0, 1]$ is a strictly increasing continuous map and fixes 0 and 1, then each G_t is surjective and injective and monotone, hence a homeomorphism. Then G is an isotopy from f to $\mathrm{id}_{[0,1]}$.

<u>Likewise</u>, if $f \in \mathrm{Homeo}([0, 1])$ is decreasingm then it is isotopic to the map that swaps 0 and 1 (given by $r(s) := 1 - s$). □

Lemma. $\mathrm{id}_{[0,1]}$ is not isotopic to r.

proof. Suppose $\{G_t \mid [0, 1] \to [0, 1]\}$ is an isotopy from $G_0 = \mathrm{id}_{[0,1]}$ to $G_1 = r$. Each G_t fixes or swaps 0 or 1. $\forall t \in [0, 1]$. $G_{t(0)} \in \{0, 1\}$. Define $\gamma(t) := G_{t(0)}$ is a path in $\{0, 1\}$ with

$$\gamma(0) = G_0(0) = 0 \text{ and } \gamma(1) = G_1(0) = 1$$

then $\{0, 1\}$ is path connected, contradiction. □

Definition (Mapping class group).

$$\mathrm{MCG}(X) := \frac{\mathrm{Homeo}(X)}{\sim}$$

where $\sim$ identify two homeomorphisms if they are isotopic forms a group called the **mapping class group** of X.

Remark.

$$\mathrm{MCG}([0,1]) \cong \mathbb{Z}_2$$

likewise, if $X = (0,1)$ or $X = \mathbb{R}$, then

$$\mathrm{Homeo}(X) = \{\text{strictly monotone bijection } f : X \to X\}$$

and

$$\mathrm{MCG}(X) \cong \mathbb{Z}_2$$

Corollary. Every homeomorphism $f : (0,1) \to (0,1)$ extends to a homeomorphism $\tilde{f} : [0,1] \to [0,1]$ defined by:

$$\tilde{f}|_{\{0,1\}} = \mathrm{id}_{\{0,1\}} \text{ if } f \text{ increasing}$$
$$\tilde{f}|_{\{0,1\}} = r|_{\{0,1\}} \text{ if } f \text{ decreasing}$$

proof. $\tilde{f}$ defined as above is a monotone bijection and hence a homeomorphism. □

2.4.2. Homeomorphism of S^1

Lemma. $\forall f \in \mathrm{Homeo}(S^1).\ \exists \tilde{f} \in \mathrm{Homeo}(\mathbb{R}).$

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\tilde{f}"] \arrow[d, "p"] & \mathbb{R} \arrow[d, "p"] \\
S^1 \arrow[r, "f"] & S^1
\end{tikzcd}$$

commutes, where $p(x) := e^{2\pi i x} \in S^1 \subseteq \mathbb{C}$. Moreover $\tilde{f}$ is unique up to

$$\tilde{f} \rightsquigarrow \tilde{f} + n \text{ for } n \in \mathbb{Z}$$

proof.

- Existence of $\tilde{f}$

  Can assume WLOG that f fixes 1, then we have

$$\begin{tikzcd}
S^1 \arrow[r, "f \cong"] & S^1 \\
S^1 \setminus \{1\} \arrow[r, "f|_{\dots} \cong"] \arrow[u, hook] & S^1 \setminus \{1\} \arrow[u, hook] \\
(0,1) \arrow[r, "g \cong"] \arrow[u, "p \cong"] \arrow[d, hook] & (0,1) \arrow[u, "p \cong"'] \arrow[d, hook] \\
{[0,1]} \arrow[r, "\tilde{g} \cong"] & {[0,1]}
\end{tikzcd}$$

We can assume WLOG that $\tilde{g}$ is increasing. Now define $\tilde{f} : \mathbb{R} \to \mathbb{R}$ by

$$\tilde{f}(x) := \tilde{g}(x - \lfloor x \rfloor) + \lfloor x \rfloor$$

can check that $\tilde{f}$ is a homeomorphism with $p \circ \tilde{f} = f \circ p$.

- Uniqueness of $\tilde{f}$ up to $\tilde{f} \rightsquigarrow \tilde{f} + n$

  Suppose $\tilde{f}, \tilde{g} : \mathbb{R} \to \mathbb{R}$ are two homeomorphisms that make the diagram commute. Then

$$\begin{aligned}
&p \circ \tilde{f} = p \circ \tilde{g} \\
\Longrightarrow\ &\forall x \in \mathbb{R}.\ \tilde{g}(x) = \tilde{f}(x) + n_x \text{ for } n_x \in \mathbb{Z} \\
\Longrightarrow\ &\tilde{g} - \tilde{f} \in \mathbb{Z} \\
\Longrightarrow\ &\tilde{g} - \tilde{f} \text{ must be constant because every constant map } \mathbb{R} \to \mathbb{Z} \text{ is a homeomorphism}
\end{aligned}$$

□

Remark. If g is increasing, then $\tilde{f}$ satisfies

$$\forall m \in \mathbb{Z}.\ \tilde{f}(x + m) = \tilde{x} + m$$

Definition. Call $f \in \mathrm{Homeo}(S^1)$
- **orientation preserving** if $\tilde{f}$ is increasing
- **orientation reversing** if $\tilde{f}$ is decreasing

Lemma. If $f \in \mathrm{Homeo}(S^1)$ is orientation preserving, then it is isotopic to id_{S^1}.

proof. Let $\tilde{f} : \mathbb{R} \to \mathbb{R}$ be a lift of f and define

$$G : \mathbb{R} \times I \to \mathbb{R}$$

by $G(x,t) := (1-t)\tilde{f}(x) + tx$. Then

1. G is isotopy from $G_0 = \tilde{f}$ to $G_1 = \mathrm{id}_{\mathbb{R}}$ (to prove this, use that each $G(-,t)$ is a continuous strictly monotonous surjection)
2. G satisfies

$$\forall m \in \mathbb{Z}.\ G(x+m,t) = G(x,t) + m$$

   (again use that $\tilde{f}$ is increasing)

Then consider following diagram

$$\begin{tikzcd}
\mathbb{R} \times I \arrow[r, "G"] \arrow[d, "p \times \mathrm{id}_I"'] & \mathbb{R} \arrow[d, "p"] \\
S^1 \times I \arrow[r, dotted, "\text{continuous } \overline{G}"'] & S^1
\end{tikzcd}$$

Then $\overline{G}$ is an isotopy from f to id_{S^1}. Likewise, if f is orientation reversing, then it is isotopic to $r(z) := \overline{z}$ for $z \in S^1 \subseteq \mathbb{C}$. $\square$

Lemma. id_{s^1} is not isotopic to r.

proof. Suppose $G : S^1 \times I \to S^1$ is an isotopy from $G_0 = \mathrm{id}_{S^1}$ to $G_1 = r$. Define

$$B = i, C = -1, A = 1$$
$$v_t := G_{t(B)} - G_{t(A)}$$
$$w_t := G_{t(C)} - G_{t(A)}$$
$$\gamma(t) := \{\text{z-coordinate of } v_t \times w_t\} \in \mathbb{R} \setminus \{0\}$$

then γ is a path in $\mathbb{R} \setminus \{0\}$ and $\gamma(0) > 0$ and $\gamma(1) < 0$, contradiction. $\square$

Remark. $\mathrm{MCG}(S^1) \cong \mathbb{Z}_2$

Definition.

$$\begin{aligned}
\mathrm{Homeo}^+(S^1) &= \{\text{orientation preserving homeomorphisms } f : S^1 \to S^1\} \\
&= [\mathrm{id}_{S^1}] < \mathrm{Homeo}(S^1)
\end{aligned}$$

Definition (Arc)**.** A proper subset $I \subset S^1$ will be called an **arc** if it is path connected. Equivalently, I is an arc if it is homeomorphic to $[a,b] \subseteq \mathbb{R}$.

Lemma. $\mathrm{Homeo}^+(S^1)$ acts transitively on pairs of disjoint arcs in S^1. That is, if $I, J \subseteq S^1$ are disjoint arcs, and I', J' are another pair of disjoint arcs, then there exists $f \in \mathrm{Homeo}^+(S^1)$ such that $f(I) = I'$ and $f(J) = J'$.

proof. Can assume

$$I' = p\left(\left[0, \frac{1}{4}\right]\right), \quad J' = p\left(\left[\frac{1}{2}, \frac{3}{4}\right]\right)$$

after applying a rotation, we can then assume that the initial point (w.r.t. counterclock wise) of I is at $1 \in S^1$.

$$\implies I = p([a,b]), \quad j = p([c,d]), 0 = a < b < c < d < 1$$

can define a piecewise linear homeomorphism $\tilde{f}$ that indices a $f : S^1 \to S^1$ with $f(I) = I'$ and $f(J) = J'$ $\square$

2.5 Handle Slides

Theorem. Let M be compact surface with boundary, S^1_+, S^1_- be two components of ∂M. Then up to homeomorphism, there exists at most two ways of attaching a 2-dimensional 1-handle to M such that the sets $\{\pm 1\} \times D^2$ are attached to $S^1_\pm$.

More precisely, given

$$f, g : \{-1, 1\} \times D^1 \to S^1_+ \cup S^1_- \subseteq \partial M$$

be two embeddings whose image intersect both S^1_+ and S^1_-, then either

$$M \sqcup_f h^1 \cong M \sqcup_g h^1$$

or

$$M \sqcup_f h^1 \cong M \sqcup_{g \circ R} h^1$$

or both, where $R : \{-1, 1\} \times D^1 \to \{-1, 1\} \times D^1$ is the identity on $\{-1\} \times D^1$ and the reflection $x \longmapsto -x$ on $\{+1\} \times D^1$.

proof.

1) Can assume that f, g both map $\{-1\} \times D^1$ to S^1_- and $\{+1\} \times D^1$ to S^1_+ since there exists a homeomorphism $h : h^1 \to h^1$ that exchanges $\{-1\} \times D^1$ and $\{+1\} \times D^1$
2) Can assume that $\mathrm{im}(f) = \mathrm{im}(g)$ follows because $\mathrm{Homeo}^+(S^+)$ acts transitively on single intervals in S^1 and on disjoint pairs.

3) Can assume that

$$g^{-1} \circ f|_{\{-1\}\times D} : \{-1\} \times D^1 \to \{-1\} \times D^1$$

is increasing, since there exists a homeomorphism $h' : h^1 \to h^1$ which restricts to an orientation-reversing homeo on $\{-1\} \times D^1$

4) Can then assume that

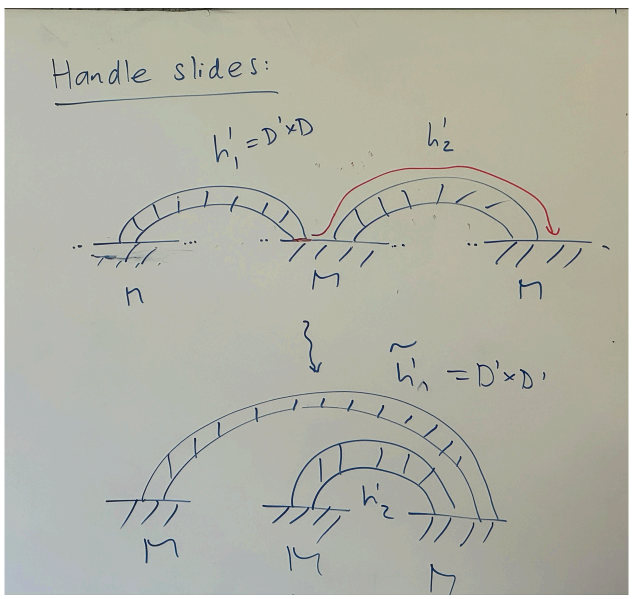
$$g^{-1} \circ f|_{\{-1\}\times D} = \mathrm{id}_{\{-1\}\times D^1}$$

since any increasing homeomorphism of $\{-1\} \times D^1$ is isotopic to $\mathrm{id}_{\{-1\}\times D^1}$, then

$$g|_{\{-1\}\times D} = f|_{\{-1\}\times D}$$

this leaves with 2 possibilities:
1) $g^{-1} \circ f \mid_{\{+1\}\times D^1}$ is increasing $\implies$ can assume $f = g$
2) $g^{-1} \circ f \mid_{\{+1\}\times D^1}$ is decreasing $\implies$ can assume $f = g \circ R$

$\square$



Note. A handle slide induces a homeomorphism

$$(M \cup h_2^1) \cup h_1^1 \to (M \cup h_2^1) \cup \widetilde{h_1^1}$$

which is the identity except in collar neighborhood of

$$\partial(M \cup h_2^1) \subseteq M \cup h_2^1$$

Remark. Handle slides can also be used to slide 1-handles off of each other

2.6 Orientations

Definition (Orientations)**.** Let S be a space homeomorphic to S^1. Then an **orientation** of S is an equivalence class of homeomorphisms $f : S^1 \to S$ where two such $f, g : S^1 \to S$ are equivalent if $g^{-1} \circ f$ is isotopic to id_{S^1}.

Definition. Let M be a 2-dimensional handle body, M_0 be union of all 0-handles. Assume all 1-handles are attached to ∂M_0 and the images of the attaching maps are pairwise disjoint. Then an **orientation on M** is a choice of orientation on the boundary of each handle in M such that for every 1-handle, the attaching map

$$f : (\partial D^1) \times \longrightarrow \partial M_0$$

has the property that $f_\pm := f|_{\{\pm 1\} \times D^1}$

Example.
- M = annulus has 2 orientations
- M = Möbius strip has no orientation

Lemma. A **connected handle body** M either admit zero or two orientations.

Theorem. M is non-orientable iff the Möbius strip can be embedded into M

Example. S^2, T orientable, P, K non-orientable.

Definition (Boundary Connected Sum)**.** M, N connected surfaces with $\partial M \neq \emptyset, \partial N \neq \emptyset$,

$$f_+ : \{+1\} \times D^1 \longrightarrow \partial M$$
$$f_- : \{-1\} \times D^1 \longrightarrow \partial N$$

be two embeddings, then the **boundary connected sum** of M and N is the surface

$$M \natural N := M \sqcup_{f_1} (D^1 \times D^1) \sqcup_{f_2} N$$

Remark. Up to homeomorphism, $M \natural N$ does not depend on the choice of f_+ and f_-

proof.
- If M, N are connected, then $\mathrm{Homeo}(M)$ and $\mathrm{Homeo}(N)$ act transitively on the components of ∂M and ∂N, respectively.
- If S is component of ∂M and ∂Nm then $\mathrm{Homeo}^+(S)$ acts transitively on intervals in S.

- If S is a component of ∂M where M is a compact surface, then there exists a homeomorphism $h : M \to M$ which sends S to itself and restricts to an orientation-reversing homeomorphism of S.

□

Definition (Connected Sum)**.** Let M, N be connected surface, possible without boundary, then the **connected sum** of M and $N$is the surface

$$M \sharp N := M_{(1)} \sqcup_f N_{(1)}$$

where f is a homeomorphism

$$\left(\partial M_{(1)}\right) \setminus \partial M \longrightarrow \left(\partial N_{(1)}\right) \setminus \partial N$$

Remark.

$$\begin{aligned} M \sharp N &\cong \left(M_{(1)} \cup N_{(1)}\right) \cup \text{cylinder} \\ &= \left(M_{(1)} \cup N_{(1)}\right) \cup (L^1 \cup h^2) \\ &= \left(M_{(1)} \natural N_{(1)}\right) \cup h^2 \end{aligned}$$

In particular,

$$(M \sharp N)_{(1)} = M_{(1)} \natural N_{(1)}$$

Example. $D^2 \natural D^2 \cong D^2$

In general, for M compact space with ∂M,

$$M \natural D^2 \cong M$$

Lemma. $P \sharp P \cong K$ where P is the projective plane and K is the Klein bottle.

proof.
1. $K = M \cup M' \cong P_{(1)} \cup P_{(1)} = P \sharp P$
2. $K_{(1)} \cong P_{(1)} \natural P_{(1)} = (P \sharp P)_{(1)} \Longrightarrow K \cong P \sharp P$

□

Lemma (Fundamental Lemma of Surface Theory)**.**

$$T \sharp P \cong K \sharp P \cong P \sharp P \sharp P$$

where T is the torus, P is the projective plane, and K is the Klein bottle.

Theorem (Classification Theorem)**.** Every closed nonempty connected surface M is homeomorphic to exactly one of the following:

1. M orientable
$$T^{(g)} = S^2 \sharp \underbrace{T \sharp \ldots \sharp T}_{g} (\text{with } g \geq 0)$$
2. M non-orientable
$$P^{(h)} = \underbrace{P \sharp \ldots \sharp P}_{h} (\text{with } h \geq 1)$$

Notation:

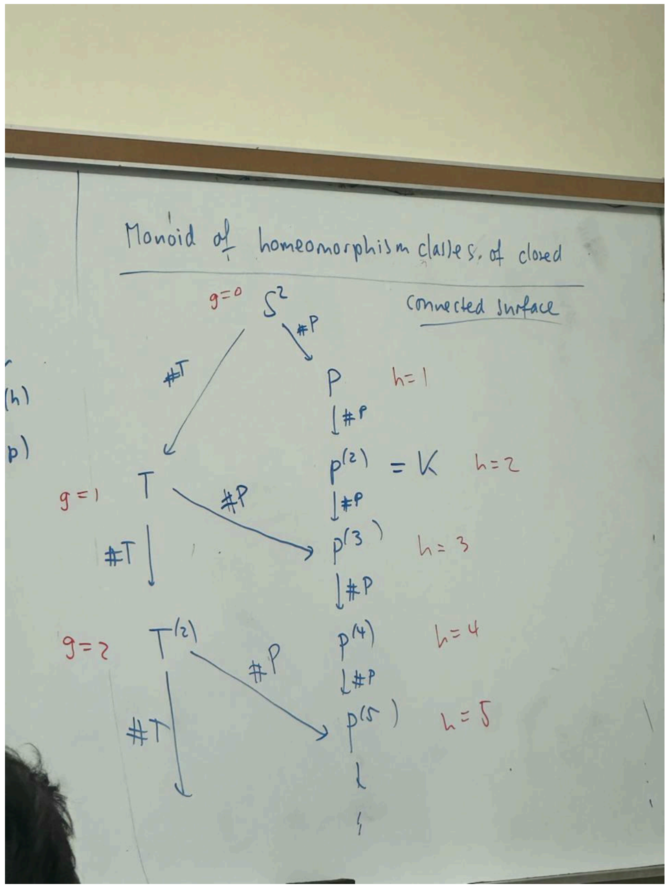
$$T^{(g)}_{(p)} = T^{(g)} - \{p \text{ open disks with disjoint closures}\}$$
$$P^{(h)}_{(p)} = P^{(h)} - \{p \text{ open disks with disjoint closures}\}$$

Corollary. Every nonempty compact connected surface M with $p \geq 0$ boundary components is homeomorphic to exactly one of the following:

1. $T^{(g)}_{(p)}$, with $g \geq 0$
2. $P^{(h)}_{(p)}$, with $h \geq 1$

Definition.

- g is the **genus** of $T^{(g)}_{(p)}$
- h is the **non-orientable genus** or **crosscap number** of $P^{(h)}_{(p)}$



Monoid of homeomorphism classes of closed connected surface

Now prove the Classification Theorem.

proof.

1. By Rados's Theorem, can assume M is a handle body
2. If M has more than one 0-handle, then there must a 1-handle h' connecting two distinct 0-handles, h_1^0, h_2^0

   $\implies$ $h_1^0 \cup h^1 \cup h_2^0 \cong D^2$
   $\implies$ can replace $h_1^0 \cup h^1 \cup h_2^0$ be a single 0-handle
   $\implies$ can reduce the number of zero handles
   $\implies$ can assume M ahs only one 0-handle

3. Can restrict to the case where M has no 2-handles because attacing 2-handle is unique up to homeomorphism
4. We may assume

$$M = h^0 \cup (h_1^1 \cup ... h_n^1)$$

Now use induction on k, the number of 1-handles.

> **Fact.** If M is a compact connected surface, then every permutation of the components of ∂N can be realized by a homeomorphism of N. (Follows from Disk Lemma)

- Base case: if $k = 0$, then $M = h^0 \cong D^2 = S^2_{(1)} = T^{(0)}_{(1)}$
- Inductive step: Assume $k > 0$, and let

$$N := h^0 \cup \left(h^1 \cup \ldots \cup h^1_{k-1}\right)$$

<u>Case 1</u>: M orientable and h^1_k is attached to a single component of ∂N. Can assume

$$\begin{aligned} M &\cong N \natural \text{ annulus} \\ &\cong T^{(g)}_{(p)} \natural S^2_{(2)} \\ &\cong T^{(g+1)}_{(p)} \end{aligned}$$

<u>Case 2</u>: M orientable and h^1_k is attached to two distinct componenets of ∂N, then ∂N has at least 2 componenets. By induction hypothesis,

$$\begin{aligned} N &\cong T^{(g)}_{(p)} \\ &\cong T^{(g)}_{(p-1)} \natural S^2_{(2)} \\ &\cong T^{(g)}_{(p-1)} \natural T_{(1)} \\ &\cong T^{(g+1)}_{(p-1)} \end{aligned}$$

<u>Case 3</u>: M is non-orientable and h^1_k is attached to a single boundary component.

$$M = N \natural \text{ annulus} \cong P^{(h)}_{(p)} \natural S^2_{(2)} \cong P^{(h)}_{(p+1)}$$

or

$$M \cong N \natural P_{(1)} = P^{(h)}_{(p)} \natural P_{(1)} = P^{(h+1)}_{(p)}$$

or

$$M \cong T^{(g)}_{(p)} \natural P_{(1)} \cong P^{(2g+1)}_{(p)}$$

<u>Case 4</u>: M is non-orientable and h^1_k is attached to two distinct components of ∂N, then ∂N has at least 2 components, $N \cong N' \natural S^2_{(2)}$, then

$$M \cong N' \natural T_{(1)} \cong P^{(h)}_{(p)} \natural T_{(1)} \cong P^{(h+2)}_{(p)}$$

or

$$M \cong N' \natural K_{(1)} = T^{(g)}_{(p)} \natural K_{(1)} = P^{(2g+2)}_{(p)}$$

□

Chapter 3

The Fundamental Group

Definition (Pointed Space). A **pointed space** is a pair (X, x_0) where X is a topological space and the "basepoint" $x_0 \in X$ is a point. The fundamental group is a topological invaraint for pointed spaces.

3.1 Homotopies

Recall. A **path** in X is a continuous map

$$f : [0,1] \longrightarrow X$$

Concatenation of Paths:

$$f, g : [0,1] \longrightarrow x \text{ paths with } f(1) = g(0)$$

forms new path

$$(f \star g)(s) := \begin{cases} f(2s) & \text{if } s \in \left[0, \frac{1}{2}\right] \\ g(2s-1) & \text{if } s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Definition (Homotopy). $f, g : X \longrightarrow Y$ be two continuous maps. A **homotopy** from f to g is a continuous map

$$H : X \times I \longrightarrow Y$$

such that $H(-,0) = f$ and $H(-,1) = g$. If such H exists, then f and g are called **homotopic** denoted $f \simeq g$.

Note. Can regard equivalence

$$\{H_t := H(-,t)\}$$

as a continuous family of continuous maps $H_t : X \longrightarrow Y$.

$$\begin{tikzcd}
X \arrow[d, hook, "x \mapsto (x,1)"'] \arrow[dr, "g"] & \\
X \times I \arrow[r, dotted, "H"] & Y \\
X \arrow[u, hook, "x \mapsto (x,0)"] \arrow[ur, "h"'] &
\end{tikzcd}$$

Definition (Path Homotopies). $f, g : I \longrightarrow X$ paths with $f(0) = g(0)$ and $g(1) = g(1)$. f and g are **path homotopic** if there exists a homotopy

$$H : I \times I \longrightarrow X$$

from f to g such that for all $t \in I$,

$$H(0,t) = f(0) = g(0)$$
$$H(1,t) = f(1) = g(1)$$

denoted

$$f \simeq_p g$$

which is an equivalence relation on paths in X. And

$$[f] : \{g \mid g \text{ a path in } X \text{ with } g \simeq_p f\}$$

is the **path homotopy class** of f.

Proposition. Suppose $f(1) = g(0)$, if $f' \simeq_p f$ and $g' \simeq_p g$, then $f' * g' \simeq_p f * g$.

proof. Choose path homotopy

$$F : I \times I \longrightarrow X \text{ from } f' \text{ to } f \qquad G : I \times I \longrightarrow X \text{ from } g' \text{ to } g$$

then we can define a path homotopy from $f' * g'$ to $f * g$:

$$(F * G)(s,t) := \begin{cases} F(2s,t) & \text{if } s \in \left[0, \frac{1}{2}\right] \\ G(2s-1,t) & \text{if } s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

$\square$

Definition (Loop)**.** Let (X, x_0) be pointed space. A **loop** in X based at x_0 is a path

$$f : I \longrightarrow X$$

such that $f(0) = f(1) = x_0$

Note.

$$\{\text{loops } f : I \longrightarrow X \text{ based at } x_0\} \overset{1:1}{\longleftrightarrow} \left\{\text{continuous map } \overline{f} : \left(\frac{I}{\partial I}, \frac{\partial I}{\partial I}\right) \to (x, x_0)\right\}$$

Definition (The Fundamental Group)**.** The **Fundamental Group** of (X, x_0) is the set

$$\begin{aligned}\Pi_1(X, x_0) &:= \frac{\{\text{loops in } X \text{ based on } x_0\}}{\text{path homotopy}} \\ &= \{[g] \mid g \text{ a loop in } X \text{ based at } x_0\}\end{aligned}$$

with concatenation of loops defines a binary operation on $\Pi_1(X, x_0)$:

$$\forall [f], [g] \in \Pi_1(X, x_0).\ [f] \mathbin{\overline{*}} [g] := [f * g]$$

Theorem. $\Pi_1(X, x_0)$ is a group with this operation.

proof.

1. $\overline{*}$ is associative

   Let $[f], [g], [h] \in \Pi_1(X, x_0)$. WTS:

   $$\begin{aligned}(f * g) * h &\simeq_p f * (g * h) \\ [0,1] = [f, g, h, h] &\qquad [f, f, g, h]\end{aligned}$$

   then

   $$(f * g) * h = (f * (g * h)) \circ k$$

   where $k : [0,1] \to [0,1]$ is the PL homeomorphism given by

   $$\begin{aligned}\left[0, \frac{1}{4}\right] &\longrightarrow \left[0, \frac{1}{2}\right] \\ \left[\frac{1}{4}, \frac{1}{2}\right] &\longrightarrow \left[\frac{1}{2}, \frac{3}{4}\right] \\ \left[\frac{1}{2}, 1\right] &\longrightarrow \left[\frac{3}{4}, 1\right]\end{aligned}$$

   and $k \simeq \mathrm{id}_{[0,1]}$ via a homotopy that fixes 0 and 1, e.g.

   $$k_{t(s)} := (1 - t)k(s) + ts$$

follows that

$$\begin{aligned}(f * g) * h &\simeq_p (f * (g * h)) \circ \mathrm{id}_{[0,1]} \\ &= f * (g * h)\end{aligned}$$

2. Π_1 has an identify element

   Let $e_{x_0} : I \to X$ be the constant path given by

   $$\forall s \in I.\ e_{x_0}(s) := x_0$$

   Claim. $\forall [f] \in \Pi_1(X, x_0).\ f * e_{x_0} \simeq_o f \simeq_p e_{x_0} * f$

   proof. Construct

   $$H(s,t) := \begin{cases} x_0 & \text{if } t \leq 2s - 1 \\ f\left(2\frac{s}{t+1}\right) & \text{if } t \geq 2s - 1 \end{cases}$$

   □

3. Π_1 has a inverses

   Let $[f] \in \Pi_1(X, x_0)$ and

   $$\forall s \in I.\ \overline{f}(s) := f(1 - s)$$

   Claim. $f * \overline{f} \simeq_p e_{x_0} \simeq_p \overline{f} * f$

   proof. Construct

   $$H(s,t) := \begin{cases} f(2s(1-t)) & \text{if } s \leq \frac{1}{2} \\ \overline{f}((2s-1)(1-t)+t) & \text{if } s \geq \frac{1}{2} \end{cases}$$

   is a path homotopy from $f * \overline{f}$ to e_{x_0}. Similarly can construct one from $\overline{f} * f$ to e_{x_0}. □

Therefore $\Pi_1(X, x_0)$ forms a group. □

Theorem (Induced Maps)**.** Let $f : (X, x_0) \longrightarrow (Y, y_0)$ be a continuous map with $f(x_0) = y_0$. The **induced map** is

$$\begin{aligned} f_* : \Pi_1(X, x_0) &\longrightarrow \Pi_1(Y, y_0) \\ [p] &\longmapsto [f \circ p] \end{aligned}$$

such that

1. f_* is well-defined
2. f_* is a group homomorphism:

   $$f_*([p][q]) = f_*([p]) f_*([q])$$

3. $\left(\mathrm{id}_{\Pi_1(X, x_0)}\right)_* = \mathrm{id}_{\Pi_1(X, x_0)}$
4. $(f \circ g)_* = f_* \circ g_*$

proof.

1. If $H : I \times I \longrightarrow X$ is a path homotopy between p_0 and p_1, then $f \circ H : I \times I \longrightarrow Y$ is a path homotopy between $f \circ p_0$ and $f \circ p_1$, that is, $[f \circ p]$ depends only on the path homotopy class $[p]$.
2. Let $[p], [q] \in \Pi_1(X, x_0)$, then

$$\begin{aligned}
(f \circ (p * q))(s) &= f((p * q)(s)) \\
&= \begin{cases} f(p(2s)) & \text{if } s \in \left[0, \frac{1}{2}\right] \\ f(q(2s-1)) & \text{if } s \in \left[\frac{1}{2}, 1\right] \end{cases} \\
&= ((f \circ p) * (f \circ q))(s) \\
\implies f \circ (p * q) &= (f \circ p) * (f \circ q) \\
\implies f_*([p][q]) &= f_*([p]) f_*([q])
\end{aligned}$$

3. Follows from the definition
4. $$\begin{aligned}
(f \circ g)_*([p]) &= [(f \circ g) \circ p] = [f \circ (g \circ p)] \\
&= f_*([g \circ p]) = f_*(g_*([p])) \\
&= (f_* \circ g_*)([p]) \\
\implies (f \circ g)_* &= f_* \circ g_*
\end{aligned}$$

$\square$

Remark. Homeomorphic pointed spaces have isomorphic fundamental groups.

Corollary. If f is a homeomorphism, then f_* is group isomorphism.

proof.

$$\begin{aligned}
f : (X, x_0) &\to (Y, y_0) \text{ homeo} \\
\implies f^{-1} : (Y, y_0) &\to (X, x_0)
\end{aligned}$$

$$\begin{aligned}
\left(f^{-1}\right)_* \circ f_* = \left(f^{-1} \circ f\right)_* = \left(\mathrm{id}_{X, x_0}\right)_* = \mathrm{id}_{\Pi_1(X, x_0)} \\
f_* \circ \left(f^{-1}\right)_* = \left(f \circ f^{-1}\right)_* = \left(\mathrm{id}_{Y, y_0}\right)_* = \mathrm{id}_{\Pi_1(Y, y_0)}
\end{aligned}$$

hence f_* and $\left(f^{-1}\right)_*$ are inverse of each other, and the isomorphism class of $\Pi_1(X, x_0)$ is a topological invariant for pointed spaces. $\square$

Example. $\Pi_1(\mathbb{R}^n, x_0) = \left\{\left[e_{x_0}\right]\right\}$

<u>Reason</u>: Any loop $f : [0, 1] \to \mathbb{R}^n$ based at $x_0 \in \mathbb{R}^n$ is path homotopic to e_{x_0} via "Straight line homotopy":

$$f_{t(s)} := (1-t) f(s) + t x_0$$

Example. $X \subseteq \mathbb{R}^n$ convex, $x_0 \in X$, then $\Pi_1(X, x_0) = \left\{\left[e_{x_0}\right]\right\}$, proof same as before.

Definition (Simply connected)**.** X is **simply connected** if it is path connected and

$$\forall x_0 \in X.\ \Pi_1(X, x_0) = \{[e]\}$$

and is independent of the choice of x_0 since x is path connected.

Remark. Any convex subspace of $\mathbb{R}^n$ is simply connected.

Example. $\Pi_1(S^1, x_0) \cong (\mathbb{Z}, +)$

Specifically, let $\omega_n : [0, 1] \longrightarrow S^1$ be the loop

$$\omega_{n(s)} := e^{2\pi i n S}$$

when $n > 0$, ω_n turns counterclockwise for n loops; when $n < 0$, ω_n turns clockwise for $-n$ loops. Then the map

$$\begin{aligned} \mathbb{Z} &\longrightarrow \Pi_1(S^1, 1) \\ n &\longmapsto [\omega_n] \end{aligned}$$

is an isomorphism.

Theorem. $\Pi_1(X \times Y, (x_0, y_0)) \cong \underbrace{\Pi_1(X, x_0) \times \Pi_1(Y, y_0)}_{\substack{\text{Direct product of group with} \\ \text{component-wise multiplication}}}$

proof. Let $p_X : X \times Y \to X, p_Y : X \times Y \to Y$ be the projections. Then the isomorphism

$$\Pi_1(X \times Y, (x_0, y_0)) \longrightarrow \Pi_1(X, x_0) \times \Pi_1(Y, y_0)$$

is given by

$$[f] \longmapsto \big((p_X)_*([f]), (p_Y)_*([f])\big)$$

inverse:

$$([f_1], [f_2]) \longmapsto [(f_1, f_2)]$$

□

Example. $\Pi_1(T) = \Pi_1(S^1 \times S^1) \cong \Pi_1(S^1) \times \Pi_1(S^1) = \mathbb{Z} \times \mathbb{Z} \cong \mathbb{Z}^2$

Theorem. $\Pi_1(S^n, x_0) = \{e\}$ for $n \geq 2$

proof.

Lemma. For $n \geq 2$, every loop p in S^n based at $x_0 \in S^n$ is path homotopic to a loop q that misses $-x_0$, the antipode of x_0.

proof. Let $p : [0, 1] \longrightarrow S^n$ be a loop based at x_0 and assume WLOG that $x_0 = (0, \ldots, 0, 1) \in S^n$, the north pole, and $x_0 = (0, \ldots, 0, -1) \in S^n$, the south pole. Let the open south hemisphere be

$$V := S^n \cap (\mathbb{R}^n \times (-\infty, 0))$$

and

$$U := p^{-1}(V) \subseteq [0, 1]$$

so U is an open subset of $(0, 1) \subseteq [0, 1]$ and $U \supseteq p^{-1}(-x_0)$, follows that U is a countable union of disjoint open intervals $I_\alpha \subseteq [0, 1]$. The I_α form an open cover for $p^{-1}(-x_0)$, so it's closed and compact, meaning there exist a finite subcover

$$\left\{I_{\alpha_1}, \ldots, I_{\alpha_k}\right\}$$

since the I_α are disjoint, none of the I_α where $\alpha \neq \alpha_1, \ldots, \alpha_k$ contain points of $p^{-1}(-x_0)$, hence

$$-x_0 \notin p\left([0, 1] \setminus \bigcup_{i=1}^{k} I_{\alpha_i}\right)$$

it's enough to show that each $p|_{\overline{I_{\alpha_i}}}$ is path-homotopic to a path q_i that misses $-x_0$. Let $I := I_{\alpha_i}$ for some i and write

$$I = (a, b) \text{ for } 0 < a < b < 1$$

then

$$\begin{aligned} p\left(\overline{I}\right) &= p([a, b]) \subseteq \overline{p((a, b))} = \overline{p(I)} \\ &\subseteq \overline{V} \\ \Longrightarrow p(a), p(b) &\in \partial V = S^{n-1} = S^n \cap (\mathbb{R}^n \times \{0\}) \end{aligned}$$

after applying a homeomorphism, we can regard $p|_{[a,b]}$ as a path in

$$D^n \cong \overline{V}$$

with endpoints in $\partial D^n = S^{n-1}$. Moreover, since $n > 2$, S^{n-1} is path connected, there exists a path q_i in $\partial D^n = S^{n-1}$ from $p(a)$ to $p(b)$. Finally, $q_i \simeq_p p|_{[a,b]}$ via a straightline homotopy in the convex set $D^n \subseteq \mathbb{R}^n$ and q_i misses the point $0 \in D^n$, which corresponds to the point $x_0 \in \overline{V} \cong D^n$. $\square$

Let $[p] \in \Pi_1(S^n, x_0)$ for $n \geq z$. By lemma, we can assume

$$\operatorname{im}(p) \subseteq S^n = \{-x_0\} \cong \mathbb{R}^n$$

then $p \simeq_p e_{x_0}$, meaning $[p] = \left[e_{x_0}\right]$ and

$$\Pi_1(S^n, x_0) = \left\{\left[e_{x_0}\right]\right\}$$

$\square$

Remark. S^n is simply connected if $n \geq 2$.

Fact (Poincaré Conjecture; shown by Perelman)**.** Every closed simply-connected 3-manifold $M \neq \emptyset$ is isomorphic to S^3.

Also true for 2-manifolds:

Fact. Every closed simply-connected 2-manifold $M \neq \emptyset$ is isomorphic to S^2.

But not true for n-manifolds with $n \geq 4$.

Example. $S^2 \times S^2$ is simply-connected, but not homeomorphic to S^4.

3.2 Fundamental Group of S^1

Theorem. For $n \in \mathbb{Z}$, let

$$\begin{aligned} \omega_n : [0,1] &\longrightarrow S^1 \\ s &\longmapsto e^{2\pi i n s} \end{aligned}$$

Then

$$\begin{aligned} \Phi : \mathbb{Z} &\longrightarrow \Pi_1(S) \\ n &\longmapsto [\omega_n] \end{aligned}$$

is a group isomorphism.

proof. First show that Φ is a homomorphism. NTS: $\Phi(m+n) = \Phi(m) + \Phi(n)$ or $[\omega_{m+n}] = [\omega_m * w_n]$. Note that

$$\begin{aligned} \omega_{m+n}(s) &= e^{2\pi i (m+n) s} \\ \omega_m(s) &= e^{2\pi i m s} \\ \omega_n(s) &= e^{2\pi i n s} = e^{2\pi i (m + ns)} \end{aligned}$$

define

$$\begin{aligned} \theta : [0,1] &\longrightarrow \mathbb{R} \\ s &\longmapsto \begin{cases} 2sm & \text{if } s \leq \frac{1}{2} \\ m + (2s-1) & \text{if } s \geq \frac{1}{2} \end{cases} \end{aligned}$$

then θ is a continuous path in $\mathbb{R}$ from 0 to $m+n$ and

$$(\omega_m * \omega_n)(s) = e^{2\pi i \theta(s)}$$

Now prove that $\Phi(n) := [\omega_n]$ is a bijection:

Let

$$\begin{aligned} q : \mathbb{R} &\longrightarrow S^1 \\ s &\longmapsto e^{2\pi i s} \end{aligned}$$

Fact. q is a covering map

Definition. Given a continuous map $f : Y \to S^1$. A lift of f through q is a continuous map $\tilde{f} : Y \longrightarrow \mathbb{R}$ such that $q \circ \tilde{f} = f$. The diagram

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "q"] & s \arrow[d, mapsto] \\ Y \arrow[ur, dotted, "\tilde{f}"] \arrow[r, "f"] & S^1 & e^{2\pi i s} \end{tikzcd}$$

Commutes.

Example. $\omega_n(s) = e^{2\pi i s} = q(ns) \in S^1$, then $\widetilde{\omega_n}(s) := ns \in \mathbb{R}$ is a lift of ω_n through $q : \mathbb{R} \longrightarrow S^1$.

Lemma (Unique Path Lifting Property, UPLP)**.** If $p : I \to S^1$ is a path and $\widetilde{x_0} \in q^{-1}(p(0))$, then there exists a unique lift $\tilde{p} : I \to \mathbb{R}$ of p through q such that $\tilde{p}(0) = \widetilde{x_0}$.

proof. $q : \mathbb{R} \to S^1$ given by $q(s) := e^{2\pi i s}$, $p : [0,1] \to S^1$ path with $x_0 := p(0)$, $\widetilde{x_0} \in q^{-1}(x_0)$. WTS: there exists a unique path $\tilde{p} : [0,1] \to \mathbb{R}$ such that $\tilde{p}(0) = \widetilde{x_0}$ and $q \circ \tilde{p} = p$. Assume WLOG that $x_0 = 1 \in S^1$. Then $q^{-1}(x_0) = q^{-1}(1) = \mathbb{Z} \subseteq \mathbb{R}$. Can assume WLOG that $\widetilde{x_0} := 0 \in \mathbb{R}$. Write

$$S^1 = U \cup V$$

where $U = S^1 \setminus \{1\}$ and $V = S^1 \setminus \{-1\}$, then

$$q^{-1}(U) = \mathbb{R} \setminus q^{-1}(1) = \mathbb{R} \setminus \mathbb{Z} = \bigsqcup_{k \in \mathbb{Z}} (k, k+1)$$

$$q^{-1}(V) = \mathbb{R} \setminus q^{-1}(-1) = \mathbb{R} \setminus \frac{1}{2} + \mathbb{Z} = \bigsqcup_{k \in \mathbb{Z}} \left(k - \frac{1}{2}, k + \frac{1}{2}\right)$$

Now let $p : [0,1] \longrightarrow S^1$ be a path with $p(0) = 1 =: x_0$. Then $p^{-1}(U) \cup p^{-1}(V)$ is an open cover of $[0,1]$, which has Lebesgue number $\delta > 0$ for this cover. If we choose $n > \frac{1}{\delta}$ then for $i = 1, \dots, n$, each $\left[\frac{i-1}{n}, \frac{1}{n}\right]$ is in $p^{-1}(U)$ and in $p^{-1}(V)$. p maps each $\left[\frac{i-1}{n}, \frac{1}{n}\right]$ to U or V or both, we will show that

$$\forall i = 0, ..., n.\ \exists \text{ a unique lift } \tilde{p}_i \text{ of } p|_{[0,\frac{i}{n}]}.\ \tilde{p}(0) = 0 \in \mathbb{R}$$

induct on i:
- $i = 0$: Define $\tilde{p_0}(0) := 0 \in \mathbb{R}$
- $i > 0$: Assume we have already constructed the lift $\widetilde{p_{i-1}}$ of $p|_{[0,\frac{i-1}{n}]}$. By construction,

$$p\left(\left[\frac{i-1}{n}, \frac{i}{n}\right]\right) \subseteq U \text{ or} \subseteq V$$

  for simplicity, assume

$$p\left(\left[\frac{i-1}{n}, \frac{i}{n}\right]\right) \subseteq U$$

  let $k \in \mathbb{Z}$ be such that

$$\widetilde{p_{i-1}}\left(\frac{i-1}{n}\right) \in (k, k+1) \subseteq q^{-1}(U)$$

  Note: q restricts to homeomorphism $(k, k+1) \to U$ so we can define

$$\tilde{p}_i := \begin{cases} \widetilde{p_{i-1}} & \text{on } \left[0, \frac{i-1}{n}\right] \\ \left(q|_{(k,k+1)}\right)^{-1} \circ p|_{\left[\frac{i-1}{n}, \frac{1}{n}\right]} & \text{on } \left[\frac{i-1}{n}, \frac{i}{n}\right] \end{cases}$$

  Easy to see: $\tilde{p}_i$ is continuous and

$$q \circ \tilde{p}_i = p|_{[0,\frac{i}{n}]}$$

  then $\tilde{p}_i$ is a lift of $p|_{[0,\frac{i}{n}]}$ through q.

Uniqueness: Suppose $\overline{p_i}$ is another lift of $p|_{[0,\frac{i}{n}]}$:
- On $\left[0, \frac{i-1}{n}\right]$, induction implies $\tilde{p}_i = \overline{p_i}$
- On $\left[\frac{i-1}{n}, \frac{i}{n}\right]$, the lifting property implies

$$q \circ \overline{p_i} = q \circ \tilde{p}_i$$

  Moreover, $\tilde{p}_i$ and $\overline{p_i}$ both map $\left[\frac{i-1}{n}, \frac{i}{n}\right]$ to $(k, k+1)$ (can be seen since they agree at $\frac{i-1}{n}$ and since $\overline{p_i}$ must map $\left[\frac{i-1}{n}, \frac{i}{n}\right]$ to a path components of $q^{-1}(U)$, follows that $\overline{p_i} = \tilde{p}_i$ on $\left[\frac{i-1}{n}, \frac{i}{n}\right]$ because q is injective on $(k, k+1)$.

□

Lemma (Path Homotopy Lifting Property, PHLP). If $H : I \times I \to S^1$ is a homotopy and $\widetilde{H_0}$ is a lift through q of $H|_{I\times\{0\}}$, then there exists a lift $\tilde{H}$ of H through q such that $\tilde{H}|_{I\times\{0\}} = \widetilde{H_0}$. Moreover, if H is a path homotopy, so is $\tilde{H}$.

proof. WTS: There exists a lift $\tilde{H} : I \times X \to \mathbb{R}$ of H extending $\widetilde{H_0}$. To define $\tilde{H}$, divide $I \times I$ into squares of the form

$$I_{ij} := \left[\frac{i-1}{n}, \frac{i}{n}\right] \times \left[\frac{j-1}{n}, \frac{j}{n}\right]$$

where n is large enough so that each $H(I_{ij})$ is in U or in V. For each $i = 1, ..., n$, use a local inverse of $q : \mathbb{R} \to S^1$ to extend the given lift $\widetilde{H_0}$ to lift $\widetilde{H_{i,1}}$ of $H|_{I_{i,1}}$.
Note: $I_{i,1} \cap I_{i+1,1} = \{i\} \times \left[0, \frac{1}{n}\right] \cong [0,1]$
$\Longrightarrow$ $\widetilde{H_{i,j}}, \widetilde{H_{i+1,j}}$ must agree on $I_{i,1} \cap I_{i+1,1}$ by the UPLP.
$\Longrightarrow$ By the piecing lemma, we obtain a well-defined lift of $H|_{I\times[0,\frac{1}{n}]}$. Now proceed inductively to fill up the square, get a lift $\tilde{H}$ of H.

Exercise: If H is a path homotopy, then so is $\tilde{H}$. □

Remark. Given
- path p_0, p_1 in S^1 with $p_0 \simeq_p p_0$
- lifts $\tilde{p_0}, \tilde{p_1}$ through q with $\tilde{p_0}(0) = \tilde{p_1}(0)$

Then $\tilde{p_0} \simeq_p \tilde{p_1}$. In particular, $\tilde{p_0}(1) = \tilde{p_1}(1)$.

Now suppose $\Phi(m) = \Phi(n)$

$\Longrightarrow$ $[\omega_m] = [\omega_n]$
$\Longrightarrow$ $\omega_m \simeq_p \omega_n$
$\Longrightarrow$ By "PHLP" $\widetilde{\omega_m} \simeq_p \widetilde{\omega_n}$ where $\widetilde{\omega_m}(s) = ms, \widetilde{\omega_n}(s) = ns$
$\Longrightarrow$ $\widetilde{\omega_m}(1) = \widetilde{\omega_n}(1)$
$\Longrightarrow$ $m = n$

so Φ is injective. Now let $[p] \in \Pi_1(S^1, 1)$, then p is a loop in S^1 based $x_0 = 1 \in S^1$. Let $\tilde{p} : I \to \mathbb{R}$ be the lift of p staring at $\widetilde{x_0} := 0 \in \mathbb{R}$,

$\Longrightarrow$ $q \circ \tilde{p} = p$ (since $\tilde{p}$ is a lift)
$\Longrightarrow$ $q(\tilde{p}(1)) = q^{-1}(1) = \mathbb{Z}$
$\Longrightarrow$ $\tilde{p}(1) \in q^{-1}(1) = \mathbb{Z}$
$\Longrightarrow$ $\tilde{p}(1) = n$ for an integer $n \in \mathbb{Z}$
$\Longrightarrow$ $\tilde{p} \wedge \widetilde{\omega_n}$ are both paths in $\mathbb{R}$ from 0 to n
$\Longrightarrow$ $\tilde{H}(s,t) := (1-t)\tilde{p}(s) + t\widetilde{\omega_n}(s) \in \mathbb{R}$ is a path homotopy from $\tilde{p}$ to $\widetilde{\omega_n}$
$\Longrightarrow$ $q \circ \tilde{H}$ is a path homotopy from ω_n
$\Longrightarrow$ $p \simeq_p \omega_n$
$\Longrightarrow$ $[p] = [\omega_n] = \Phi(n)$
$\Longrightarrow$ $[p] \subseteq \mathrm{im}(\Phi(n))$

So Φ is surjective, hence an isomorphism. □

Fact. $p : I \longrightarrow S^1$ path, $\widetilde{x_0} \in q^{-1}(p(0))$, then there exists a unique lift $\tilde{p} : I \longrightarrow \mathbb{R}$ of p through q such that $\tilde{p}(0) = \widetilde{x_0}$

3.3 Dependence on the base point

X space, $x_0, x_1 \in X$ be points in same path component. Let $\alpha : [0,1] \to X$ be path from x_0 to x_1. Can define a map

$$\begin{aligned} \alpha_* : \Pi_1(X, x_1) &\longrightarrow \Pi_1(X, x_0) \\ [p] &\longmapsto [\alpha * p * \overline{\alpha}] \end{aligned}$$

where p is a loop based at x_1.

Fact.
- α_* is well-defined ($[\alpha * p * \overline{\alpha}]$ depends only on $[p]$)
- α_* is an isomorphism with inverse $(\alpha_*)^{-1} = (\overline{\alpha})_*$
- If α, β are composable paths, then $(\alpha * \beta)_* = \alpha_* \circ \beta_*$
- α_* depends only on $[\alpha]$

So: If X is path connected, then the isomorphism class of $\Pi_1(X, x_0)$ is independent of the choice of x_0.

Proposition. Let $f, g : A \to B$ be homotopic continuous maps, with homotopy $F : A \times I \to B$. For $a_0 \in A$, let

$$\alpha(t) := F(a_0, t)$$

then the following commutes:

$$\begin{tikzcd} & \Pi_{B,g(b_0)} \arrow[dd, "\alpha_*"] \\ \Pi_{A,g(a_0)} \arrow[ur, "g_*"] \arrow[dr, "f_*"'] & \\ & \Pi_{B,f(b_0)} \end{tikzcd}$$

proof. See book, page 228. $\square$

3.4 Homotopy invariance of Π_1

Definition (Homotopy Equivalence). X, Y space, $f : X \to Y$ continuous. f is a **homotopy equivalence** if there exists a continuous map $g : Y \to X$ such that

$$g \circ f \simeq \mathrm{id}_X \text{ and } f \circ g \simeq \mathrm{id}_Y$$

In this case, g is called a **homotopy inverse** of f and X and Y are called **homotopy equivalent**, denoted

$$X \simeq Y$$

Example. Every homeomorphism is a homotopy equivalence.

Definition (Contractible). X is Contractible if

$$X \simeq \{1 \text{ point}\}$$

Easy to see:

$$X \text{ Contractible} \iff \mathrm{id}_X \simeq c_{x_0}$$

where

$$\begin{aligned} c_{x_0} : X &\longrightarrow X \\ x &\longmapsto x_0 \end{aligned}$$

Example. $\mathbb{R}^n$ is contractible because

$$\mathrm{id}_{\mathbb{R}^n} \simeq c_0$$

via the homotopy

$$H(x,t) := (1-t)x \text{ where } x \in \mathbb{R}^n, t \in [0,1]$$

Like wise, every convex $A \subseteq \mathbb{R}^n \neq \emptyset$ is contractible.

Example.

$$\mathbb{R}^2 - \{0\} \cong S^1 \times (0, \infty) \simeq S^1 \times \{1 \text{ point}\} \cong S^1$$

Likewise

$$\mathbb{R}^n - \{0\} \simeq S^{n-1}$$

Theorem. If $f : X \to Y$ is a homotopy equivalence, then

$$f_* : \Pi_1(X, x_0) \longrightarrow \Pi_1(Y, f(x_0))$$

is an isomorphism for any $x_0 \in X$.

proof. Let $g : Y \to X$ be homotopy inverse for f. Consider four fundamental groups:

$$
\begin{tikzcd}
 & & & \Pi_1(Y, f(x_0)) \arrow[d, "\cong"', "\beta_*"] \\
\Pi_1(X, x_0) \arrow[r, "f_*"] \arrow[drr, "\mathrm{id}_{\Pi_1}(X, x_0)"'] & \Pi_1(X, f(x_0)) \arrow[r, "g_*"] \arrow[urr, "\mathrm{id}_{\Pi_1}(X, f(x_0))"] & \Pi_1(X, g(f(x_0))) \arrow[r, "f'_*"] & \Pi_1(X, f(g(f(x_0)))) \\
 & & \Pi_1(X, x_0) \arrow[u, "\alpha_*"] &
\end{tikzcd}
$$

Then

$$
\begin{aligned}
g_* \circ f_* = \underbrace{\alpha_*}_{\text{iso}} &\implies g_* \text{ injective} \\
f_* \circ g_* = \underbrace{\beta_*}_{\text{iso}} &\implies f_* \text{ injective} \\
g_* \text{ invertible} \implies g_* \circ f_* = \alpha_* &\implies f_* \text{ invertible}
\end{aligned}
$$

$\square$

3.5 Degree

Definition (Degree). $f : S^1 \longrightarrow S^1$ continuous. Consider

$$
\begin{tikzcd}
{[0,1]} \arrow[r, "q|_{[0,1]}"] \arrow[rr, bend right, "f' := f \circ g|_{[0,1]}"'] & S^1 \arrow[r, "f"] & S^1
\end{tikzcd}
$$

Let $\tilde{f}' : [0,1] \to \mathbb{R}$ be a lift of f' through g, then the **degree** of f is defined by

$$\deg(f) := \tilde{f}'(1) - \tilde{f}'(0)$$

Note. This independent of the chosen lift $\tilde{f}'$ of f' because any two lifts differ by

$$\tilde{f}' \rightsquigarrow \tilde{f}' + n \text{ for } n \in \mathbb{Z}$$

Proposition. if $f, g : S^1 \longrightarrow S^1$ are homotopic continuous maps, then

$$\deg(f) = \deg(g)$$

proof. Let $F : S^1 \times I \longrightarrow S^1$ be a homotopy from f and g. Define

$$F' : I \times I \longrightarrow S'$$

by $F' := F \circ (g|_I \times \mathrm{id}_I)$. Let $\tilde{f}'$ be lift of f' and $\tilde{F}'$ be a lift of F' extending $\tilde{f}'$.

$$\begin{tikzcd}
(0,1) \arrow[r, "\tilde{g}'"] & (1,1) \\
(0,0) \arrow[u, "\tilde{h}"] \arrow[r, "\tilde{f}'"] & (1,0) \arrow[u, "\tilde{h}+n, n \in \mathbb{Z}"']
\end{tikzcd}$$

By definition of degree:

$$\begin{aligned}
\deg(f) &= \tilde{F}'(1,0) - \tilde{F}'(0,0) \\
\deg(g) &= \tilde{F}'(1,1) - \tilde{F}'(0,1) \\
\deg(f) - \deg(g) &= \tilde{F}'(1,0) - \tilde{F}'(0,0) - \big(\tilde{F}'(1,1) - \tilde{F}'(0,1)\big) \\
&= \tilde{h}(0) + n - \tilde{h}(1) + n - \big(\tilde{h}(0) - \tilde{h}(1)\big) \\
&= 0 \Longrightarrow \deg(f) = \deg(g)
\end{aligned}$$

□

Example. Use this to show

$$\deg(f) = n \Longleftrightarrow f \simeq \text{the map } z \in S^1 \longmapsto z^n \in S'$$

Corollary. if $f, g : S^1 \longrightarrow S^1$ continuous, then

$$\deg(f \circ g) = \deg(f)\deg(g)$$

proof. Let $m := \deg(f)$, $n := \deg(g)$, then

$$\begin{gathered}
f \simeq z^m \text{ and } g \simeq z^n \\
f \circ g \simeq (z^n)^m = z^{nm} \\
\deg(f \circ g) = nm = mn = \deg(f)\deg(g)
\end{gathered}$$

□

Corollary. if $f : S^1 \longrightarrow S^1$ is a homomorphism, then

$$\deg(f) = \pm 1$$

in fact, if f is orientation preserving then $\deg(f) = 1$; if f is orientation reversing then $\deg(f) = -1$.

proof.

$$\begin{aligned}\deg(f)\deg(g) &= \deg(f \circ f^{-1}) \\ &= \deg(\mathrm{id}_{S^1}) \\ &= \deg(z^1) \\ &= 1\end{aligned}$$

hence $\deg(f) \in \mathbb{Z}^X = \{\pm 1\}$. □

Proposition. If $f : S^1 \longrightarrow S^1$ extends to a continuous map $F : D^2 \longrightarrow S^1$, then

$$\deg(f) = 0$$

proof. Follows because in this case

$$f \simeq \text{constant map} \simeq (z \longmapsto z^0)$$

since D^2 is contractible. □

Note. Recall that

$$\begin{aligned}\mathbb{R}^2 - \{0\} &\cong S^1 \\ \Longrightarrow \Pi_1(\mathbb{R}^2 - \{0\}) &\cong \Pi_1(S^1) \cong \mathbb{Z}\end{aligned}$$

can also be see as follows:

$$\begin{aligned}\Pi_1(\mathbb{R}^2 - \{0\}) &\cong \Pi_1(S^1 \times (0,\infty)) \\ &\cong \Pi_1(S^1) \times \Pi_1((0,\infty)) \\ &\cong \Pi_1(S^1) \cong \mathbb{Z}\end{aligned}$$

but $\Pi_1(\mathbb{R}^n - \{0\}) = \Pi_1(S^{n-1}) = 0$ for $n > 2$

Definition. Let $f : S^1 \longrightarrow \mathbb{R}^2 - \{0\}$ be continuous, can define $\tilde{f} : S^1 \longrightarrow S^1$ by

$$\tilde{f}(x) := \frac{f(x)}{\|f(x)\|}$$

can define

$$\underbrace{\deg(f)}_{\text{"winding number"}} := \deg\left(\tilde{f}\right)$$

intuitively, how many times f wind around 0.

3.5.1. Applications

Definition (Retraction). Let X space, $A \subseteq X$ subspace. A **retraction** from X to A is a continuous map $r : X \longrightarrow A$ such that

$$r|_A = \mathrm{id}_A$$

Theorem. There exists no retraction $r : D^2 \longrightarrow S^1$.

proof. Suppose such retraction r exists, then

$$\deg(r|_{S^1}) = \deg(\mathrm{id}_{S^1}) = 1$$

but also $r|_{S^1}$ extends to the entire D^2, namely r, so $\deg(r|_{S^1}) = 0$. Contradiction. □

Theorem (Fundamental Theorem of Algebra). Let

$$P(z) = z^n + a_{n-1}z^{n-1} + \dots + a_0$$

be a complex polynomial with $n > 0$. Then P has a zero in $\mathbb{C}$.

proof. Let

$$M := \max\{|a_0|, \dots, |a_{n-1}|\}$$

and choose $k \geq 1$. Then for $z \in kS^1$, the circle around 0 of radius k,

$$P(z) = z^n\left(1 + \frac{a_{n-1}}{z} + \dots + \frac{a_0}{z^n}\right) = z^n(1 + b(z)) \neq 0$$

where $|b(z)| < \frac{1}{2}$ since $z \in kS^1$. So $f := P|_{kS^1}$ is a map

$$f : kS^1 \longrightarrow \mathbb{C} - \{0\}$$

Moreover, f is homotopic to $z^n|_{kS^1}$ via

$$H(z,t) := z^n(1 + (1-t)b(z)) \neq 0$$
$$\implies \deg(f) = \deg(z^n|_{kS^1}) = n > 0$$

Now suppose P has no zeroes. Then P takes values in $\mathbb{C} - \{0\}$, so f extends to the map

$$P|_{kD^2} : kD^2 \longrightarrow \mathbb{C} - \{0\} \implies \deg(f) = 0$$

Contradiction. □

Remark. Suppose $0 < k_1 < k_2$ are such that

$$\deg\left(P|_{k_1S^1}\right) \neq \deg\left(P|_{k_2S^1}\right)$$

Then P must have a zero in $\{z \in \mathbb{C} \mid k_1 < |z| < k_2\}$.

Definition. $f : I \to S^1$ continuous such that $f(1) = -f(0)$. Can define

$$\deg(f) := \tilde{f}(1) - \tilde{f}(0) \in \frac{1}{2} + \mathbb{Z}$$

where $\tilde{f} : I \longrightarrow \mathbb{R}$ is a lift of f.

Theorem. There exists no continuous map $f : S^2 \to S^1$ such that $\forall x \in S^2.\ f(-x) = -f(x)$

proof. Suppose such an f exists, then

$$\deg(f|_{S^1}) = 0$$

since $f|_{S^1}$ extends to the northern or southern hemisphere. But

$$S^1 = I_+ \cup I_-$$

then

$$\begin{aligned}\deg(f|_{S^1}) &= \deg\left(f|_{I_+}\right) + \deg\left(f|_{I_-}\right) \\ &= 2\deg\left(f|_{I_+}\right) \text{ since } f(-x) = -f(x) \\ &= 2\left(n + \frac{1}{2}\right) \\ &= 2n + 1 \neq 0\end{aligned}$$

□

Theorem (Brouwer)**.** Every continuous map $f : D^2 \longrightarrow D^2$ has a fixed point x with $f(x) = x$.

proof. Suppose f has no fixed point, and define

$$g : D^2 \longrightarrow \partial D^2$$

with $g(x)$ be the intersection of line $xf(x)$, one can check g is continuous and $g|_{\partial D^2} = \mathrm{id}_{\partial D^2}$, follows that g is a retraction from D^2 to ∂D^2. By no-retraction theorem, contradiction. □

Note. Not true if D^2 is replaced by $D^2 \setminus \partial D^2 \cong \mathbb{R}^2$.

3.6 Seifert-van Kampen Theorem

Definition (Word)**.** Let G_1, G_2 be groups. A **word** in G_1 and G_2 is a finite sequence

$$(w_1, w_2, ..., w_n) \text{ where } w_i \in G_1 \text{ or } G_2$$

Definition (Free Product). The **free product** of G_1 and G_2 is the set

$$G_1 * G_2 := \frac{\{\text{words in } G_1 \text{ and } G_2\}}{\sim}$$

where $\sim$ is generated by:

- If w_i and w_{i+1} belong to the same group, then

$$(\ldots, w_i, w_{i+1}, \ldots) \sim (\ldots, w_i w_{i+1}, \ldots)$$

- If w_i is the identity element of G_1 or G_2, then

$$(\ldots, w_{i-1}, w_i, w_{i+1}, \ldots) \sim (\ldots, w_{i-1}, w_{i+1}, \ldots)$$

Note. $G_1 * G_2$ is a group with multiplication given by concatenation.

Theorem (Univeersal Property). Given homomorphism $\varphi_i : G_i \longrightarrow H$, $i = 1, 2$, there exists a unique homomorphism

$$\varphi : G_1 * G_2 \longrightarrow H$$

that extends φ_1 and φ_2:

$$\begin{tikzcd}
G_1 \arrow[dr, "i_1"] \arrow[drr, "\varphi_1"] & & \\
& G_1 * G_2 \arrow[r, dashed, "\exists!\varphi"] & H \\
G_1 \arrow[ur, "i_2"] \arrow[urr, "\varphi_2"'] & &
\end{tikzcd}$$

where $i_j(w) := [(w)]$, $w \in G_j$.

Example.

$$F_K := \underbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}_{k<\infty} = \mathbb{Z}^{*k}$$

is a **free group** on k generators.

Proposition. Any group H with $k < \infty$ generators can be written as

$$H \cong \frac{F_k}{N}$$

for a normal subgroup $N \trianglelefteq F_k$.

proof. Let $h_1, ..., h_k$ be generators of H, and define

$$\begin{aligned} \Phi_i : \mathbb{Z} &\longrightarrow H \\ n &\longmapsto (h_i)^n \end{aligned}$$

The Φ_i induce a surjective map $\varphi : \mathbb{Z}^{*k} \longrightarrow H$. Define $N := \ker(\varphi)$, then

$$H \cong \frac{\mathbb{Z}^{*k}}{\ker(\varphi)} = \frac{F_k}{N}$$

□

Remark. N is the **set of relations**.

Definition (Group Presentation)**.** If H is finitely generators $h_1, ..., h_j$, then

$$H \cong \langle h_1, ..., h_k \mid N \rangle$$

If N is also finitely generated as a normal subgroup of F_k by elements $r_1, ..., r_l$, then write

$$H \cong \langle h_1, ..., h_k \mid r_1, ..., r_k \rangle$$

In this case, H is finitely presented.

Note. Iff $k_1 \neq k_2$, then

$$F_{k_1} \ncong F_{k_2}$$

proof. Follows because the abelianizations of F_{k_1} and F_{k_2} are $\mathbb{Z}^{k_1}$ and $\mathbb{Z}^{k_2}$ and

$$\mathbb{Z}^{k_1} \ncong \mathbb{Z}^{k_2}$$

□

Theorem (Seifert-van Kampen)**.** Let X be space, $X = A \cup B$ where $A, B \subseteq X$ are open and $A \cap B$ is path-connected. Pick $x_0 \in A \cap B$ and consider

$$\begin{tikzcd} \Pi_1(A \cap B, x_0) \arrow[r, "\psi_A"] \arrow[d, "\psi_B"'] & \Pi_1(A, x_0) \arrow[d, "\varphi_A"] \\ \Pi_1(B, x_0) \arrow[r, "\varphi_B"] & \Pi_1(X, x_0) \end{tikzcd}$$

Where all maps are induced by inclusions. Then the homeomorphism equation

$$\varphi : \Pi_1(A, x_0) * \Pi_1(B, x_0) \longrightarrow \Pi_1(X, x_0)$$

induced by φ_A and φ_B is surjective and

$$\ker(\varphi) = \left\{ \begin{matrix} \text{the smallest normal subgroup} \\ \text{containing all } \psi_A(\gamma)\psi_B(\gamma)^{-1} \\ \text{for all } \gamma \in \Pi_1(A \cap B, x_0) \end{matrix} \right\}$$

Note. So:

$$\Pi_1(X, x_0) \cong \frac{\Pi_1(A, x_0) * \Pi_1(B, x_0)}{(\forall \gamma \in \Pi_1(A \cap B, x_0)\psi_A(\gamma) = \psi_B(\gamma).}$$

<u>Special cases</u>:
1. X, A, B as before. If $A \cap B$ is simply connected, then

$$\Pi_1(X, x_0) \cong \Pi_1(A, x_0) * \Pi_1(B, x_0)$$

<u>Application</u>

$(X, x_0), (Y, y_0)$ pointed spaces, then the "wedge sum" of X and Y

$$X \vee Y := \frac{X \sqcup Y}{x_0 \sim y_0}$$

Let $z_0 := \overline{x_0} = \overline{y_0} \in X \vee Y$ and suppose:
- x_0, y_0 are closed in X, Y, respectively
- x_0, y_9 have open neighborhoods in X, Y, respectively, which deformation retract to x_0, y_0, then

$$\Pi(X \vee Y, z_0) \cong \Pi_1(X, x_0) * \Pi_1(Y, y_0)$$

Definition (Deformation retraction)**.** A **deformation retraction** of X to $A \subseteq X$ is a homotopy

$$H : X \times X \longrightarrow X$$

such that
- $\forall x \in X.\ H(x, 0) = x$
- $\forall x \in X.\ H(x, 1) \in A$
- $\forall x \in A.\ \forall t \in I.\ H(x, t) = x$

Example.
- $\Pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}$. More generally:

$$\Pi_1(S^1 \vee S^1 \vee \ldots \vee S^1) \cong \mathbb{Z}^{*k} = F_k$$

- X, A, B as in SvK. If A and B are simply connected, then so is X. Follows because

$$\Pi_1(X, x_0) \cong \frac{\Pi_1(A, x_0) * \Pi_1(B, x_0)}{\ldots} = 0 * 0 = 0$$

can be used to show that $\Pi_1(S^n) = 0$ for $n > 1$.

- X, A, B as in SvK, and $\Pi_1(B, x_0) = 0$. Then

$$\Pi_1(X, x_0) \cong \frac{\Pi_1(A, x_0)}{N}$$

where N is the normal subgroup generated by the image of

$$\psi_A : \Pi_1(A \cap B, x_0) \longrightarrow \Pi_1(A, x_0)$$

Example . $X = P = M \cup D^2$. $A = \text{nbhd}(M) \subseteq P$ and $B = \text{nbhd}(D^2) \subseteq P$, then $\Pi_1(B, x_0) = 0$. Follows

$$\Pi_X = \frac{\Pi_1(M)}{\Pi_1(\partial M)} = \frac{\mathbb{Z}\langle\gamma\rangle}{2\gamma = 0} = \frac{\mathbb{Z}}{2\mathbb{Z}}$$

3.7 Fundatmental groups of surfaces

3.7.1. Surfaces with $\partial M \neq \emptyset$

Claim.

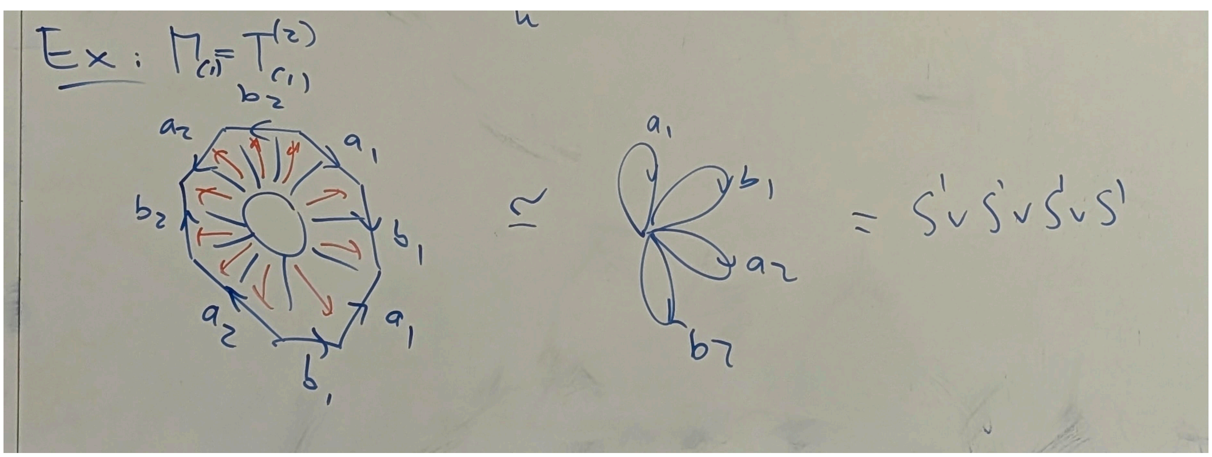
$$\Pi_1\left(T^{(g)}_{(1)}\right) \cong \mathbb{Z}^{2g} = F_{2g}$$

$$\Pi_1\left(P^{(h)}_{(1)}\right) \cong \mathbb{Z}^{h} = F_{h}$$

proof. Start with a closed surface $M = T^{(g)}$ or $M = P^{(k)}$ and realize it as a polygon with identified edges. Poke a hold in the middle of the polygon to get $M_{(1)}$. The result is homotopic to

$$\underbrace{S^1 \cup ... \cup S^1}_{k} \text{ where } k = 2g \text{ or } k = h$$

Example.



$\square$

3.7.2. Surface without boundary

Ley M be closed connected surface. Poke a hold and put it back in.

$$M = A \cup B \text{ where } A = \operatorname{nbhd}\left(M_{(1)}\right) \text{ and } B = \operatorname{nbhd}(D^2)$$

follows that

$$\begin{aligned}\Pi_1(M) &\cong \Pi_1\left(M_{(1)}\right) * \overbrace{\Pi_1(D^2)}^{=0} \\ &= \frac{\Pi_1\left(M_{(1)}\right)}{\Pi_1\left(\partial M_{(1)}\right)} = \frac{F_k}{N}\end{aligned}$$

where N is generated by the image in $\Pi_1\left(M_{(1)}\right)$ of $\Pi_1\left(\partial M_{(1)}\right)$.

$M = T^{(g)}$

$$\Pi_1(M) = \left\langle a_1, b_1, ..., a_g, b_g \,\middle|\, \sum_{i=1}^{g} [a_i, b_i] \right\rangle$$

where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$ be the commutator.

$MP^{(h)}$

$$\Pi_1(M) = \left\langle a_1, ..., a_n \mid a_{1_1^2} a_2^2 ... a_n^2 \right\rangle$$

Definition (Abelianization Π_1)**.** Let (X, x_0) path connected.

$$\Pi_1(X, x_0)_{\text{ab}} := \frac{\Pi_1(X, x_0)}{N}$$

where N is the group generated by all commutators.

Definition (First Betti Number)**.** Suppose X is path connected and $x_0 \in X$, $\Pi(X, x_0)_{\text{ab}}$ has finite rank, then the **first Betti number** of X is

$$b_1(X) := \operatorname{rank} \Pi_1(X, x_0)_{\text{ab}}$$

Theorem. M a connected 2-dim handlebody with $\partial M \neq \varphi$, then

$$X(M) = 1 - b_1(M)$$

can be shown without using that $X(M)$ is a topo invariant.